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EXERCISES

IN

WENTWORTH'S ANALYTIC GEOMETRY,

WITH SOLUTIONS.

BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1894.

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PREFACE.

THIS book is intended for teachers, *and for them only.* The publishers will sell it to no persons except teachers of Wentworth's Analytic Geometry. Every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn & Company.

It is hoped that young teachers will derive great help from studying the systematic solutions of the exercises, and that every teacher who is pressed for time will find great relief by not being obliged to work out every problem himself.

G. A. WENTWORTH.

EXETER, N.H., 1888.

TEACHERS' EDITION.

◆

Ex. 1. Page 3.

1. What are the co-ordinates of the origin? *Ans.* 0, 0.

2. In what quadrants are the following points (a and b being given lengths): $(-a, -b)$, $(-a, b)$, (a, b), $(a, -b)$.

Ans. Third, second, first, fourth.

3. To what quadrants is a point limited if its abscissa is positive? negative? ordinate positive? ordinate negative?

Ans. First and fourth, second and third, first and second, third and fourth.

4. In what line does a point lie if its abscissa $= 0$? if its ordinate $= 0$?

Ans. The axis of y; the axis of x.

5. A point (x, y) moves parallel to the axis of x; which one of its co-ordinates remains constant in value? *Ans.* The axis of y.

6. Construct or *plot* the points: $(2, 3)$, $(3, -3)$, $(-1, -3)$, $(-4, 4)$, $(3, 0)$, $(-3, 0)$, $(0, 4)$, $(0, -1)$, $(0, 0)$.

7. Construct the triangle whose vertices are the points $(2, 4)$, $(-2, 7)$, $(-6, -8)$.

8. Construct the quadrilateral whose vertices are the points $(7, 2)$, $(0, -9)$, $(-3, -1)$, $(-6, 4)$.

9. Construct the quadrilateral whose vertices are $(-3, 6)$, $(-3, 0)$, $(3, 0)$, $(3, 6)$. What kind of a quadrilateral is it?

Ans. A square whose sides are 6 each.

10. Mark the four points (2, 1), (4, 3), (2, 5), and (0, 3), and connect them by straight lines. What kind of a figure do these four lines enclose? *Ans.* A square whose sides are $2\sqrt{2}$ each.

11. The side of a square $= a$; the origin of co-ordinates is the intersection of the diagonals. What are the co-ordinates of the vertices (i.) if the axes are parallel to the sides of the square? (ii.) if the axes coincide with the diagonals?

Let $ABCD$ be the square, and O the intersection of its diagonals.

(i.) Take OX parallel to BA, and OY parallel to DA, as axes of reference. Then, by Geometry, OX and OY bisect the sides of the square, and $OH = HA = \frac{1}{2}a$. Hence the vertex A is $(\frac{1}{2}a, \frac{1}{2}a)$, B is $(-\frac{1}{2}a, \frac{1}{2}a)$, C is $(-\frac{1}{2}a, -\frac{1}{2}a)$, and D is $(\frac{1}{2}a, -\frac{1}{2}a)$.

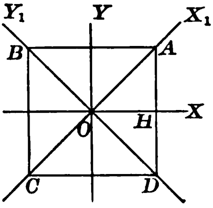

FIG. 1.

(ii.) Take the diagonals OX_1 and OY_1 as axes of reference. Then $OA = \sqrt{OH^2 + HA^2} = \sqrt{\frac{1}{4}a^2 + \frac{1}{4}a^2} = \frac{1}{2}a\sqrt{2}$, and $OA = OB = OC = OD$.

Hence the vertex A is $(\frac{1}{2}a\sqrt{2}, 0)$, B is $(0, \frac{1}{2}a\sqrt{2})$, C is $(-\frac{1}{2}a\sqrt{2}, 0)$, and D is $(0, -\frac{1}{2}a\sqrt{2})$.

12. The side of an equilateral triangle $= a$; the origin is taken at one vertex and the axis of x coincides with one side. What are the co-ordinates of the three vertices?

Let OPB be an equilateral triangle each of whose sides is a. Take OBX and OY as the axes of co-ordinates; then, evidently, the vertex O is $(0, 0)$, and the vertex B is $(a, 0)$.

Draw PH parallel to OY; then $OH = \frac{1}{2}a$, and $HP = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{1}{2}a\sqrt{3}$.

Hence the vertex P is $(\frac{1}{2}a, \frac{1}{2}a\sqrt{3})$.

13. The line joining two points is bisected at the origin. If the co-ordinates of one of the points are a and b, what are the co-ordinates of the other?

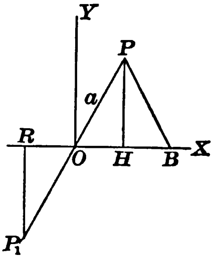

FIG. 2.

Let the line joining P and P_1 (Fig. 2) be bisected at the origin O. Draw PH and P_1R parallel to OY; then the right triangles OHP and ORP_1 will be equal. Hence $OR = -OH$, and $RP_1 = -HP$. Therefore if P be (a, b), P_1 will be $(-a, -b)$.

14. Connect the points (5, 3) and (5, –3) by a straight line. What is the direction of this line ?

Each point being at the distance 5 to the right of the axis of y, the line must be parallel to this axis.

Ex. 2. Page 5.

1. Find the value in circular measure of the angles 1°, 45°, 90°, 180°, 270°.

We have given $360° = 2\pi$.

$$\therefore 1° = \frac{\pi}{180}; \quad \therefore 45° = \frac{\pi}{4}; \quad \therefore 90° = \frac{\pi}{2}; \quad \therefore 180° = \pi; \text{ and } 270° = \tfrac{3}{2}\pi.$$

2. In circular measure, the unit angle is that angle whose arc is equal to the radius of the circle. What is the value of this angle in degrees, etc. ?

Dividing both members of the equation $2\pi = 360°$ by 2π, we obtain

$$1 = \frac{360°}{2\pi} = \frac{180°}{3.1416} = 57° \, 17' \, 45''.$$

Ex. 3. Page 7.

1. Find the distance from the point (–2, 5) to the point (–8, –3).

If d denote the distance between the two points (x_1, y_1) and (x_2, y_2),

then $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad [1]$$

Here let (x_1, y_1) be (–2, 5), and (x_2, y_2) be (–8, –3).

Substituting these values in the formula, we have

$$d = \sqrt{[(-8) - (-2)]^2 + [(-3) - 5]^2} = 10.$$

2. Find the distance from the point (1, 3) to the point (6, 15).

Here let (x_1, y_1) be (1, 3), and (x_2, y_2) be (6, 15)

Substituting these values in [1], we obtain

$$d = \sqrt{(6 - 1)^2 + (15 - 3)^2} = 13.$$

3. Find the distance from the point (–4, 5) to the point (0, 2).

Here let (x_1, y_1) be (–4, 5), and (x_2, y_2) be (0, 2).

Substituting these values in [1], we obtain

$$d = \sqrt{[0 - (-4)]^2 + (2 - 5)^2} = 5.$$

4. Find the distance from the origin to the point $(-6, -8)$.

Here let (x_1, y_1) be $(0, 0)$, and (x_2, y_2) be $(-6, -8)$.

Substituting these values in [1], we obtain

$$d = \sqrt{(-6)^2 + (-8)^2} = 10.$$

5. Find the distance from the point (a, b) to the point $(-a, -b)$.

Here let (x_1, y_1) be (a, b), and (x_2, y_2) be $(-a, -b)$

Substituting these values in [1], we obtain

$$d = \sqrt{(-a-a)^2 + (-b-b)^2} = 2\sqrt{a^2 + b^2}.$$

6. Find the lengths of the sides of a triangle if the vertices are the points $(15, -4)$, $(-9, 3)$, $(11, 24)$.

First, let (x_1, y_1) be $(15, -4)$, and (x_2, y_2) be $(-9, 3)$; then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-24)^2 + (7)^2} = 25.$$

Second, let (x_1, y_1) be $(-9, 3)$, and (x_2, y_2) be $(11, 24)$; then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(20)^2 + (21)^2} = 29.$$

Third, let (x_1, y_1) be $(11, 24)$, and (x_2, y_2) be $(15, -4)$; then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4)^2 + (-28)^2} = 20\sqrt{2}.$$

Hence the sides are 25, 29, and $20\sqrt{2}$ in length.

7. Find the lengths of the sides of a triangle if the vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.

First, let (x_1, y_1) be $(2, 3)$, and (x_2, y_2) be $(4, -5)$; then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{2^2 + (-8)^2} = 2\sqrt{17}.$$

Second, let (x_1, y_1) be $(4, -5)$, and (x_2, y_2) be $(-3, -6)$; then

$$d = \sqrt{(-7)^2 + (-1)^2} = 5\sqrt{2}.$$

Third, let (x_1, y_1) be $(-3, -6)$, and (x_2, y_2) be $(2, 3)$; then

$$d = \sqrt{(5)^2 + (9)^2} = \sqrt{106}.$$

Hence the lengths of the sides are $2\sqrt{17}$, $5\sqrt{2}$, and $\sqrt{106}$.

8. Find the lengths of the sides of a triangle if the vertices are the points $(0, 0)$, $(3, 4)$, $(-3, 4)$.

First, let (x_1, y_1) be $(0, 0)$, and (x_2, y_2) be $(3, 4)$; then

$$d = \sqrt{3^2 + 4^2} = 5.$$

Or, when one point, as (x_1, y_1), is the origin, we have

$$d = \sqrt{x_2^2 + y_2^2} = \sqrt{3^2 + 4^2} = 5.$$

Second, let (x_1, y_1) be $(3, 4)$, and (x_2, y_2) be $(-3, 4)$; then

$$d = \sqrt{(-6)^2 + 0^2} = 6.$$

Third, let (x_1, y_1) be $(-3, 4)$, and (x_2, y_2) be $(0, 0)$; then

$$d = \sqrt{x_1^2 + y_1^2} = \sqrt{(-3)^2 + 4^2} = 5.$$

Hence the lengths of the sides are 5, 6, and 5.

9. Find the lengths of the sides of a triangle if the vertices are the points $(0, 0)$, $(-a, 0)$, $(0, -b)$.

First, let (x_1, y_1) be $(0, 0)$, and (x_2, y_2) be $(-a, 0)$; then

$$d = \sqrt{(-a)^2} = a.$$

Or, since the second point is on the axis x, a to the left of the origin, the distance is a.

Second, let (x_1, y_1) be $(-a, 0)$, and (x_2, y_2) be $(0, -b)$; then

$$d = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}.$$

Third, since $(0, -b)$ is on the axis of y, b below the origin, the distance from $(0, -b)$ to $(0, 0)$ is b.

Hence the lengths of the sides are a, $\sqrt{a^2 + b^2}$, and b.

10. The vertices of a quadrilateral are $(5, 2)$, $(3, 7)$, $(-1, 4)$, $(-3, -2)$. Find the lengths of the sides, and also of the diagonals.

By constructing the quadrilateral, or by inspection, we learn that its vertices taken in order are $(5, 2)$, $(3, 7)$, $(-1, 4)$, and $(-3, -2)$. Hence to find the length of its sides, we find the distances between its adjacent vertices $(5, 2)$ and $(3, 7)$, $(3, 7)$ and $(-1, 4)$, $(-1, 4)$ and $(-3, -2)$, $(-3, -2)$ and $(5, 2)$.

Let (x_1, y_1) be $(5, 2)$, and (x_2, y_2) be $(3, 7)$; then

$$d = \sqrt{(-2)^2 + 5^2} = \sqrt{29}.$$

Let (x_1, y_1) be $(3, 7)$, and $(x_2 y_2)$ be $(-1, 4)$; then

$$d = \sqrt{(-4)^2 + (-3)^2} = 5.$$

Let (x_1, y_1) be $(-1, 4)$, and (x_2, y_2) be $(-3, -2)$; then

$$d = \sqrt{(-2)^2 + (-6)^2} = 2\sqrt{10}.$$

Let (x_1, y_1) be $(-3, -2)$, and (x_2, y_2) be $(5, 2)$; then

$$d = \sqrt{8^2 + 4^2} = 4\sqrt{5}.$$

Hence the sides in order are $\sqrt{29}$, 5, $2\sqrt{10}$, and $4\sqrt{5}$.

To find the lengths of its diagonals, we find the distances between its opposite vertices $(5, 2)$ and $(-1, 4)$, and $(3, 7)$ and $(-3, -2)$.

Let (x_1, y_1) be $(5, 2)$, and (x_2, y_2) be $(-1, 4)$; then

$$d = \sqrt{(-6)^2 + 2^2} = 2\sqrt{10}.$$

Let (x_1, y_1) be $(3, 7)$, and (x_2, y_2) be $(-3, -2)$; then

$$d = \sqrt{(-6)^2 + (-9)^2} = 3\sqrt{13}.$$

Hence the lengths of the diagonals are $2\sqrt{10}$ and $3\sqrt{13}$.

11. One end of a line whose length is 13 is the point $(-4, 8)$; the ordinate of the other end is 3. What is its abscissa?

Here we have $d = 13$, $x_1 = -4$, $y_1 = 8$, and $y_2 = 3$.
Substituting these values in [1], we obtain

$$13 = \sqrt{(x_2 + 4)^2 + (-5)^2} \qquad (1)$$

Solving (1) for x_2, we obtain

$$x_2 = 8 \text{ or } -16.$$

Hence the abscissa of the other end of the line is 8 or -16.

12. What equation must the co-ordinates of the point (x, y) satisfy if its distance from the point $(7, -2)$ is equal to 11?

Here $d = 11$, (x_1, y_1) is $(7, -2)$, and (x_2, y_2) is (x, y).
Substituting these values in [1], we obtain

$$11 = \sqrt{(x-7)^2 + (y+2)^2},$$

or

$$(x-7)^2 + (y+2)^2 = 121;$$

which is the required equation.

13. What equation expresses algebraically the fact that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$?

First, let (x_1, y_1) be $(2, 3)$, and (x_2, y_2) be (x, y); then

$$d = \sqrt{(x-2)^2 + (y-3)^2}.$$

Now let (x_1, y_1) be $(4, 5)$, and $(x_2\, y_2)$ be (x, y); then

$$d = \sqrt{(x-4)^2 + (y-5)^2}.$$

But these two values of d are to be equal; hence we have

$$\sqrt{(x-2)^2+(y-3)^2}=\sqrt{(x-4)^2+(y-5)^2}, \tag{1}$$

as the equation expressing the required condition.

Squaring (1) and simplifying it, we have

$$x+y=7,$$

as the required equation in its simplest form.

14. If the value of a quantity depends on the *square* of a length, it is immaterial whether the length be considered positive or negative. Why?

Because the square of any positive quantity is equal to the square of a numerically equal negative quantity.

Ex. 4. Page 9.

1. What are the co-ordinates of the point half-way between the points (5, 3) and (7, 9)?

If (x, y) be the point half-way between the points (x_1, y_1) and (x_2, y_2), then

$$x=\frac{x_1+x_2}{2}, \quad y=\frac{y_1+y_2}{2}. \qquad [2]$$

Here let (x_1, y_1) be (5, 3), and (x_2, y_2) be (7, 9); then

$$x=\frac{5+7}{2}=6, \quad y=\frac{3+9}{2}=6.$$

Hence the required point is (6, 6).

2. What are the co-ordinates of the point half-way between the points (−6, 2) and (4, −2)?

Here let (x_1, y_1) be (−6, 2), and (x_2, y_2) be (4, −2); then

$$x=\frac{-6+4}{2}=-1, \quad y=\frac{2-2}{2}=0.$$

Hence the required point is (−1, 0).

3. What are the co-ordinates of the point half-way between the points (5, 0) and (−1, −4)?

Here let (x_1, y_1) be (5, 0), and (x_2, y_2) be (−1, −4); then

$$x=\frac{5-1}{2}=2, \quad y=\frac{0-4}{2}=-2.$$

Hence the required point is (2, −2).

4. The vertices of a triangle are $(2,3)$, $(4,-5)$, $(-3,-6)$; find the middle points of its sides.

First let (x_1, y_1) be $(2,3)$, and (x_2, y_2) be $(4,-5)$; then

$$x=\frac{2+4}{2}=3,\quad y=\frac{3-5}{2}=-1.$$

Next let (x_1, y_1) be $(4,-5)$, and (x_2, y_2) be $(-3,-6)$; then

$$x=\frac{4-3}{2}=\frac{1}{2},\quad y=\frac{-5-6}{2}=-\frac{11}{2}.$$

Next let (x_1, y_1) be $(-3,-6)$, and (x_2, y_2) be $(2,3)$; then

$$x=\frac{-3+2}{2}=-\frac{1}{2},\quad y=\frac{-6+3}{2}=-\frac{3}{2}.$$

Hence the middle points are $(3,-1)$, $(\frac{1}{2},-\frac{11}{2})$, and $(-\frac{1}{2},-\frac{3}{2})$, respectively.

5. The middle point of a line is $(6,4)$, and one end of the line is $(5,7)$. What are the co-ordinates of the other end?

Here (x, y) is $(6,4)$, and (x_1, y_1) is $(5,7)$.

Substituting these values in [2], we obtain

$$6=\frac{5+x_2}{2},\quad 4=\frac{7+y_2}{2}. \tag{1}$$

Solving equations (1) for x_2 and y_2, we obtain

$$x_2=7,\quad y_2=1.$$

Hence the other end of the line is $(7,1)$.

6. A line is bisected at the origin; one end of the line is the point $(-a, b)$. What are the co-ordinates of the other end?

Here (x, y) is $(0,0)$, and (x_1, y_1) is $(-a, b)$.

Substituting these values in [2], we obtain

$$0=\frac{-a+x_2}{2},\quad 0=\frac{b+y_2}{2}.$$

$$\therefore x_2=a,\qquad y_2=-b.$$

Hence the other end of the line is $(a,-b)$.

7. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

Let AOB be the right triangle, right angle at O. Take OA and OB as the axes of x and y, respectively. Let $OA = a$, $OB = b$, and H be the middle point of AB; then the vertex O is $(0, 0)$, A is $(a, 0)$, B is $(0, b)$, and, by [2], H is $(\frac{1}{2}a, \frac{1}{2}b)$.

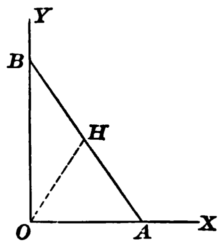

FIG. 3.

Now $\qquad AB = \sqrt{OA^2 + OB^2} = \sqrt{a^2 + b^2}.$

$\qquad AH = HB = \frac{1}{2}\sqrt{a^2 + b^2}.$

But $\qquad OH = \sqrt{(\frac{1}{2}a)^2 + (\frac{1}{2}b)^2}$

$\qquad = \frac{1}{2}\sqrt{a^2 + b^2}.$

$\therefore HO = HA = HB.$ Q.E.D.

8. Prove that the diagonals of a parallelogram mutually bisect each other.

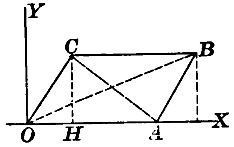

FIG. 4.

Let $OABC$ be any parallelogram. Take OX and OY as the axes of co-ordinates, and draw CH and BX parallel to OY.

Let $OH = b$, $OA = a$, and $HC = h$; then the vertex O is $(0, 0)$, A is $(a, 0)$, B is $(a+b, h)$, and C is (b, h).

Hence the middle point of the line OB is $(\frac{1}{2}[a + b], \frac{1}{2}h)$, and that of CA also is $(\frac{1}{2}[a + b], \frac{1}{2}h)$. Since their middle points coincide, they mutually bisect each other.

9. Show that the values of x and y in [2] hold true when the two given points both lie in the second quadrant.

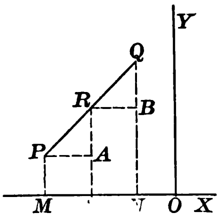

FIG. 5.

The demonstration in § 8 applies word for word to Fig. 5.

10. Solve the problem of § 9 when the line PQ is cut *externally* instead of internally, in the ratio $m : n$.

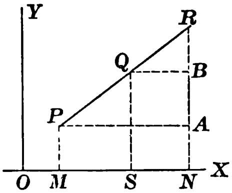

FIG. 6.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points. Let $R(x, y)$ be the required point, so that

$$PR : RQ :: m : n.$$

Draw PM, QS, RN parallel to OY, and PA, QB parallel to OX.

The right triangles PAR and QBR, being mutually equiangular, are similar; therefore

$$\frac{PA}{QB}=\frac{PR}{RQ}=\frac{m}{n}, \text{ and } \frac{AR}{BR}=\frac{PR}{RQ}=\frac{m}{n}.$$

Substituting for PA, QB, AR, BR, their values, we have

$$\frac{x-x_1}{x-x_2}=\frac{m}{n}, \text{ and } \frac{y-y_1}{y-y_2}=\frac{m}{n}.$$

Solving these equations for x and y, we obtain

$$x=\frac{mx_2-nx_1}{m-n},\quad y=\frac{my_2-ny_1}{m-n}.$$

The minus sign before n we should expect; for, while in § 9 PR and RQ extend in the same direction, here they extend in opposite directions.

11. What are the co-ordinates of the point which divides the line joining $(3,-1)$ and $(10,6)$ in the ratio $3:4$?

Here (x_1, y_1) is $(3,-1)$, (x_2, y_2) is $(10,6)$, and $m:n$ is $3:4$.

Hence $$x=\frac{mx_2+nx_1}{m+n}=\frac{30+12}{7}=6,$$

and $$y=\frac{my_2+ny_1}{m+n}=\frac{18-4}{7}=2.$$

Hence the required point is $(6,2)$.

12. The line joining $(2,3)$ and $(4,-5)$ is trisected; determine the point of trisection nearest $(2,3)$.

Here (x_1, y_1) is $(2,3)$, (x_2, y_2) is $(4,-5)$, and $m:n$ is $1:2$.

Hence $$x=\frac{mx_2+nx_1}{m+n}=\frac{4+4}{3}=\frac{8}{3},$$

and $$y=\frac{my_2+ny_1}{m+n}=\frac{-5+6}{3}=\frac{1}{3}.$$

Hence the required point is $(\frac{8}{3}, \frac{1}{3})$.

13. A line AB is produced to a point C, such that $BC=\frac{1}{2}AB$. If A and B are the points $(5,6)$ and $(7,2)$, what are the co-ordinates of C?

From the conditions, we have

$$AB:BC::2:1.\quad \therefore AC[=AB+BC]:BC::3:1.$$

That is, the line AB is divided externally at C, in the ratio $3:1$.

Here (x_1, y_1) is $(5, 6)$, (x_2, y_2) is $(7, 2)$, and $m:n$ is $3:1$. Hence, using the formulas in No. 10, we have

$$x = \frac{mx_2 - nx_1}{m - n} = \frac{21 - 5}{2} = 8,$$

and $$y = \frac{my_2 - ny_1}{m - n} = \frac{6 - 6}{2} = 0.$$

Hence C is $(8, 0)$.

14. A line AB is produced to a point C, such that $AB:BC = 4:7$. If A and B are the points $(5, 4)$ and $(6, -9)$, what are the co-ordinates of C?

If $AB:BC::4:7$, then $AC[=AB+BC]:BC::11:7$.

Hence C divides AB externally in the ratio $11:7$.

Here (x_1, y_1) is $(5, 4)$, (x_2, y_2) is $(6, -9)$, and $m:n$ is $11:7$.

Hence $$x = \frac{mx_2 - nx_1}{m - n} = \frac{66 - 35}{4} = 7\tfrac{3}{4},$$

and $$y = \frac{my_2 - ny_1}{m - n} = \frac{-99 - 28}{4} = -31\tfrac{3}{4}.$$

Hence C is $(7\tfrac{3}{4}, -31\tfrac{3}{4})$.

15. Three vertices of a parallelogram are $(1, 2)$, $(-5, -3)$, $(7, -6)$. What is the fourth vertex?

Of the triangle whose vertices are the three given points, either side may be taken as a diagonal of the parallelogram; hence there are three solutions.

(i.) If we take the line joining the points $(-5, -3)$ and $(7, -6)$ as a diagonal, we have for its mid-point

$$x = \frac{7 - 5}{2} = 1, \quad y = \frac{-6 - 3}{2} = -4\tfrac{1}{2}.$$

But as the diagonals of a parallelogram mutually bisect each other, the mid-point $(1, -4\tfrac{1}{2})$ is also the mid-point of the diagonal joining $(1, 2)$ and the fourth vertex (x_2, y_2). Hence we have

$$1 = \frac{1 + x_2}{2}, \quad -\frac{9}{2} = \frac{2 + y_2}{2}.$$

Whence $x_2 = 1$, $y_2 = -11$, and the fourth vertex is $(1, -11)$.

(ii.) If we take the line joining the points (1, 2) and (−5, −3) as a diagonal, we have for its mid-point

$$x=\frac{1-5}{2}=-2,\quad y=\frac{2-3}{2}=-\frac{1}{2}.$$

But the mid-point $(-2, -\frac{1}{2})$ is also the mid-point of the diagonal joining (7, −6) and the fourth vertex (x_2, y_2). Hence we have

$$-2=\frac{7+x_2}{2},\quad -\frac{1}{2}=\frac{-6+y_2}{2}.$$

Whence $x_2=-11$, $y_2=5$, and the fourth vertex is (−11, 5).

(iii.) Taking the line joining the points (1, 2) and (7, −6) as a diagonal, we have for its mid-point

$$x=\frac{1+7}{2}=4,\quad y=\frac{2-6}{2}=-2.$$

Hence for the fourth vertex (x_2, y_2), we have

$$4=\frac{-5+x_2}{2},\quad -2=\frac{-3+y_2}{2}.$$

Whence the fourth vertex is (13, −1).

Ex. 5. Page 13.

1. A point $P(x, y)$ revolves about the point $Q(x_1, y_1)$, keeping always at the distance a from it. Mention the constants and the variables in this case. What is the total change in the value of each variable ?

a is a constant, since it denotes a constant distance; and, since (x_1, y_1) is a fixed point, x_1 and y_1 are also constants. Since (x, y) moves about Q, x and y are both variables. $P(x, y)$ cannot move farther than the distance a to the right or to the left of Q, or above or below Q; hence the total change in either x or y is $2a$.

2. A point $Q(x, y)$ moves: first parallel to the axis of y, then parallel to the axis of x, then equally inclined to the axes. Point out in each case the constants and the variables.

(i.) When the point (x, y) moves parallel to the axis of y, y changes and is therefore a *variable*, while x does not change and is therefore a *constant.*

(ii.) When the point (x, y) moves parallel to the axis of x, x changes and is therefore a variable, while y does not change and is therefore a constant.

(iii.) In the third case both x and y change and are therefore variables.

Ex. 6. Page 20.

Determine and construct the loci of the following equations (the locus in each case being either a straight line or a circle):

1. $x - 6 = 0$.

Here $x = 6$; hence the point must move parallel to the axis of y at the distance 6 to the right of it. Whence the locus is a straight line parallel to the axis of y at the distance 6 to the right of it.

2. $x + 5 = 0$.

Here $x = -5$; hence the point must move parallel to the axis of y at the distance 5 to the left. Whence the locus is a straight line parallel to the axis of y at the distance 5 to the left of it.

3. $y = -7$.

Here the point must move parallel to the axis of x at the distance 7 below it; hence the locus is a straight line parallel to the axis of x at the distance 7 below it.

4. $x = 0$.

Here the point must move on the axis of y; hence the locus is that axis.

5. $y = 0$.

Here the point must move on the axis of x; hence the locus is that axis.

6. $x + y = 0$.

Here $y = -x$; that is, y and x are numerically equal, but unlike in sign. Hence the point must move along a line which bisects the second and fourth quadrants, which is therefore the required locus.

7. $x - 2y = 0$.

Here $y = \frac{1}{2}x$;

hence when $x = 0, 1, 2, 3, 4, \ldots\ldots,$
$y = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\ldots$

Locating the points $(0, 0)$, $(1, \frac{1}{2})$, $(2, 1)$, $(3, \frac{3}{2})$, $(4, 2)$, ..., and drawing through them a continuous line, we find that the line AB in Fig. 7 is the required locus.

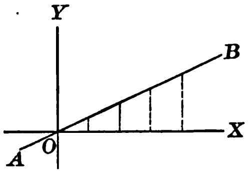

FIG. 7.

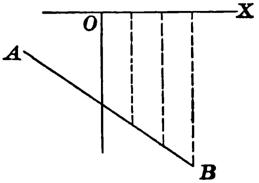

FIG. 8.

8. $2x + 3y + 10 = 0.$

Here $y = -\frac{2}{3}x - \frac{10}{3}$;

hence when $x = \quad 0, \quad 1, \quad 2, \quad 3, \ldots\ldots$

$$y = -\tfrac{10}{3}, -4, -4\tfrac{2}{3}, -5\tfrac{1}{3}, \ldots\ldots$$

Locating the points $(0, -\frac{10}{3})$, $(1, -4)$, $(2, -4\frac{2}{3})$, $(3, -5\frac{1}{3})$, ..., and drawing through them a continuous line, we find that the locus is the straight line AB.

9. $9x^2 - 25 = 0.$

This equation may be written in the form

$$(3x - 5)(3x + 5) = 0. \qquad (1)$$

Now (1) is satisfied when

$$3x - 5 = 0, \text{ or } 3x + 5 = 0. \quad \therefore x = \tfrac{5}{3}, \text{ or } x = -\tfrac{5}{3}.$$

Hence the locus consists of two straight lines parallel to the axis of y, one $\frac{5}{3}$ to the right of it, the other $\frac{5}{3}$ to the left.

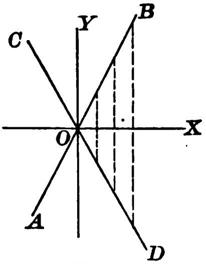

FIG. 9.

10. $4x^2 - y^2 = 0.$

This equation may be written in the form

$$(2x - y)(2x + y) = 0. \qquad (1)$$

Now (1) is satisfied when $2x - y = 0$, or $2x + y = 0$; that is,

when $\quad y = 2x, \qquad (2)$

or $\quad y = -2x. \qquad (3)$

In (2) when

$$x = 0, 1, 2, 3, 4, \ldots\ldots$$
$$y = 0, 2, 4, 6, 8, \ldots\ldots$$

Locating the points $(0, 0)$, $(1, 2)$, $(2, 4)$, $(3, 6)$, ..., and tracing a line through them, we obtain AB as the locus of (2). Similarly, we find CD to be the locus of (3). Hence the locus of $4x^2 - y^2 = 0$ consists of the two lines AB and CD

11. $x^2 - 16y^2 = 0.$

This equation may be written in the form

$$(x - 4y)(x + 4y) = 0, \qquad (1)$$

which is evidently satisfied when $x - 4y = 0$, or $x + 4y = 0$; that is,

when $y = \frac{1}{4}x$, (1)

or $y = -\frac{1}{4}x$. (2)

In (1) when

$$x = 0,\ 1,\ 2,\ 3,\ \ldots\ldots$$

$$y = 0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ \ldots\ldots$$

Locating the points $(0, 0)$, $(1, \frac{1}{4})$, $(2, \frac{1}{2})$, $(3, \frac{3}{4})$, ..., and tracing a line through them, we obtain AB as the locus of (1). Similarly, we find the locus of (2) to be CD. Hence the locus of the given equation consists of the two lines AB and CD.

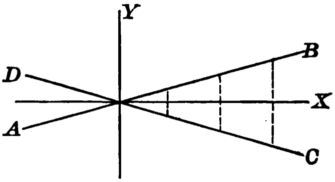

FIG. 10.

12. $x^2 + y^2 = 36$.

This equation may be written in the form

$$6^2 = (x - 0)^2 + (y - 0)^2,$$

which, from [1], states that the point (x, y) is at the distance 6 from $(0, 0)$, or the origin; hence the locus is the circumference of the circle whose centre is the origin and whose radius is 6.

13. $x^2 + y^2 - 1 = 0$.

From [1] this equation states that the point (x, y) is at the distance 1 from the origin; hence its locus is the circumference of a circle whose centre is the origin and whose radius is 1.

14. $x(y + 5) = 0$.

This equation is satisfied when $x = 0$, or $y = -5$. Hence the locus consists of the axis of y and a line parallel to, and 5 below, the axis of x.

15. $(x - 2)(x - 3) = 0$.

This equation is satisfied when $x = 2$, or $x = 3$; hence the locus consists of two lines parallel to the axis of y, one 2 and the other 3 to the right of that axis.

16. $(y - 4)(y + 1) = 0$.

This equation is satisfied when $y = 4$, or $y = -1$; hence the locus consists of two lines parallel to the axis of x, one 4 above, and the other 1 below, that axis.

17. What is the geometric meaning of the equation $5x^2-17x-12=0$?

This equation may be written in the form $5(x-4)(x+\frac{3}{5})=0$, which is evidently satisfied when $x=4$, or $x=-\frac{3}{5}$; hence the locus is two lines parallel to the axis of y, one 4 to the right, and the other $\frac{3}{5}$ to the left, of that axis.

18. What is the geometric meaning of the equation $y^2+3y=0$?

This equation may be written in the form $y(y+3)=0$, which is satisfied when $y=0$, or $y=-3$; therefore the locus consists of the axis of x and a line parallel to, and 3 below, that axis.

19. What two lines form the locus of the equation $xy+4x=0$?

This equation may be written in the form $x(y+4)=0$, which is evidently satisfied when $x=0$, or $y=-4$; hence the locus consists of the axis of y, and a line parallel to and 4 below the axis of x.

20. Is the point $(2,-5)$ situated in the locus of the equation $4x-3y-22=0$?

By trial we find that the equation is not satisfied for $x=2$ and $y=-5$; hence the point $(2,-5)$ is not in the locus of the equation.

21. Is the point $(4,-6)$ in the locus of the equation $y^2=9x$?

By trial we find that the equation is satisfied for $x=4$ and $y=-6$; hence the point $(4,-6)$ is in the locus of $y^2=9x$.

22. Is the point $(-1,-1)$ in the locus of the equation

$$16x^2+9y^2+15x-6y-18=0?$$

By trial we find that the equation is not satisfied for $x=-1$ and $y=-1$; hence the point $(-1,-1)$ is not in the locus of the given equation.

23. Does the locus of the equation $x^2+y^2=100$ pass through the point $(-6,8)$?

By trial we find that $x^2+y^2=100$ is satisfied for $x=-6$ and $y=8$; hence the locus of $x^2+y^2=100$ passes through the point $(-6,8)$.

24. Which of the loci represented by the following equations pass through the origin?

(1) $3x+2=0$.
(2) $3x-11y+7=0$.
(3) $x^2-16y^2-10=0$.
(4) $ax+by+c=0$.
(5) $3x=2y$.
(6) $3x-11y=0$.
(7) $x^2-16y^2=0$.
(8) $ax+by=0$.

Each of the loci of equations (5), (6), (7), (8) passes through the origin, since each of these equations is satisfied for $x = 0$ and $y = 0$. The locus of no one of the equations (1), (2), (3), (4) passes through the origin, since no one of these equations is satisfied for $x = 0$ and $y = 0$.

25. The abscissa of a point in the locus of the equation $3x - 4y - 7 = 0$ is 9; what is the value of the ordinate?

Putting $x = 9$ in $3x - 4y - 7 = 0$, we obtain $y = 5$; hence the ordinate of the point is 5.

26. Determine that point in the locus of $y^2 - 4x = 0$ for which the ordinate $= -6$.

Putting $y = -6$ in the equation $y^2 - 4x = 0$, we obtain $x = 9$; hence the required point is $(9, -6)$.

27. Determine the point where the line represented by the equation $7x + y - 14 = 0$ cuts the axis of x.

Putting $y = 0$ in the equation we obtain $x = 2$; hence the line cuts the axis of x at the point $(2, 0)$.

Ex. 7. Page 23.

Find the intercepts of the curve

$$4x + 3y - 48 = 0. \tag{1}$$

In (1) for $y = 0$, $x = 12$; and for $x = 0$, $y = 16$.

Hence the intercept on the axis of x is 12, and that on the axis of y is 16.

2. Find the intercepts of the curve

$$5y - 3x - 30 = 0. \tag{1}$$

In (1) for $y = 0$, $x = -10$; and for $x = 0$, $y = 6$.

Hence the intercept on the axis of x is -10, and that on the axis of y is 6.

3. Find the intercepts of the curve

$$x^2 + y^2 = 16. \tag{1}$$

In (1) for $y = 0$, $x = \pm 4$; and for $x = 0$, $y = \pm 4$.

Hence the intercepts on the axis of x are ± 4, and those on the axis of y are ± 4.

4. Find the intercepts of the curve

$$9x^2 + 4y^2 = 16. \qquad (1)$$

In (1) for $y = 0$, $x = \pm \frac{4}{3}$; and for $x = 0$, $y = \pm 2$.

Hence the intercepts on the axis of x are $\pm \frac{4}{3}$, and those on the axis of y are ± 2.

5. Find the intercepts of the curve

$$9x^2 - 4y^2 = 16. \qquad (1)$$

In (1) for $y = 0$, $x = \pm \frac{4}{3}$; and for $x = 0$, $y = \pm 2\sqrt{-1}$.

Hence the intercepts on the axis of x are $\pm \frac{4}{3}$, and the locus does not cut the axis of y.

6. Find the intercepts of the curve

$$9x^2 - 4y = 16. \qquad (1)$$

In (1) for $y = 0$, $x = \pm \frac{4}{3}$; and for $x = 0$, $y = -4$.

Hence the intercepts on the axis of x are $\pm \frac{4}{3}$, and the intercept on the axis of y is -4.

7. Find the intercepts of the curve

$$a^2x^2 + b^2y^2 = a^2b^2. \qquad (1)$$

In (1) for $y = 0$, $x = \pm b$; and for $x = 0$, $y = \pm a$.

Hence the intercepts on the axis of x are $\pm b$, and those on the axis of y are $\pm a$.

8. Find the intercepts of the curve

$$x - 3 = 0. \qquad (1)$$

Here $x = 3$; hence the intercept on the axis of x is 3. Since the locus is parallel to the axis of y, the locus does not cut that axis, and has no intercept on it.

Or we may write (1) in the form

$$x + 0y - 3 = 0. \qquad (2)$$

Now in (2) for $x = 0$, $y = 3 \div 0$, or ∞; that is, the point in which (2) cuts the axis of y has vanished by passing to infinity.

9. Find the intercepts of the curve

$$x^2 - 9 = 0.$$

Here $x = \pm 3$; hence the intercepts on the axis of x are ± 3.

10. Find the intercepts of the curve

$$x^2 - y^2 = 0.$$

Here for $y = 0$, $x = \pm 0$; for $x = 0$, $y = \pm 0$.

Hence the intercept on each axis is 0; that is, the locus passes through the origin.

11. Find the intercepts of the curve

$$y^2 = 4x.$$

Here for $y = 0$, $x = 0$; for $x = 0$, $y = \pm 0$.

Hence the locus cuts the axes only at the origin.

12. Find the intercepts of the curve

$$x^2 + y^2 - 4x - 8y = 32.$$

Here for $y = 0$, $x = 8$ and -4; for $x = 0$, $y = 4 \pm 4\sqrt{3}$.

Hence the intercepts on the axis of x are 8 and -4; while those on the axis of y are $4 + 4\sqrt{3}$ and $4 - 4\sqrt{3}$.

13. Find the intercepts of the curve

$$x^2 + y^2 - 4x - 8y = 0.$$

Here for $y = 0$, $x = 0$ and 4; for $x = 0$, $y = 0$ and 8.

Hence the intercepts on the axis of x are 0 and 4, and those on the axis of y are 0 and 8.

14. Find the intercepts of the curve

$$(x - 5)^2 + (y - 6)^2 = 20.$$

Here for $y = 0$, x is imaginary; and for $x = 0$, y is imaginary.

Hence the locus does not cut either axis.

15. Find the points of intersection of the curves

$$3x - 4y + 13 = 0, \quad 11x + 7y - 104 = 0.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 5$ and $y = 7$; hence the point of intersection of their loci is (5, 7). When these equations are considered separately, x and y in each are variables; but when they are regarded as simultaneous, x and y in each are constants, and denote the co-ordinates of the point of intersection of their loci.

16. Find the points of intersection of the curves

$$2x + 3y = 7, \quad x - y = 1.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 2$ and $y = 1$; hence the point of intersection of their loci is (2, 1).

17. Find the point of intersection of the curves

$$x - 7y + 25 = 0, \quad x^2 + y^2 = 25.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 3$, $y = 4$, and $x = -4$, $y = 3$; hence the points of intersection of their loci are $(3, 4)$ and $(-4, 3)$.

18. Find the points of intersection of the curves

$$3x + 4y = 25, \quad x^2 + y^2 = 25.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 3$ and $y = 4$; hence the point of intersection of their loci is $(3, 4)$.

19. Find the points of intersection of the curves

$$x + y = 8, \quad x^2 + y^2 = 34.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 5$, $y = 3$, and $x = 3$, $y = 5$; hence the points of intersection of their loci are $(5, 3)$ and $(3, 5)$.

20. Find the points of intersection of the curves

$$2x = y, \quad x^2 + y^2 - 10x = 0.$$

Regarding these equations as simultaneous, and solving them for x and y, we obtain $x = 0$, $y = 0$, and $x = 2$, $y = 4$; hence the points of intersection of their loci are $(0, 0)$ and $(2, 4)$.

21. The equations of the sides of a triangle are $2x + 9y + 17 = 0$, $7x - y - 38 = 0$, $x - 2y + 2 = 0$. Find the co-ordinates of its three vertices.

To find the vertex of the angle included between the loci of $2x + 9y + 17 = 0$ and $7x - y - 38 = 0$, we regard these equations as simultaneous, and solve them for x and y. We thus obtain $x = 5$ and $y = -3$; hence this vertex is the point $(5, -3)$.

Similarly, we find the vertex of the angle between the sides $7x - y - 38 = 0$ and $x - 2y + 2 = 0$, to be the point $(6, 4)$; and the vertex of the angle between the sides $x - 2y + 2 = 0$ and $2x + 9y + 17 = 0$ to be the point $(-4, -1)$.

22. The equations of the sides of a triangle are $5x + 6y = 12$, $3x - 4y = 30$, $x + 5y = 10$. Find the lengths of its sides.

This problem consists of the two parts:

(i.) Finding the three vertices of the triangle.

(ii.) Finding the distances between these vertices taken two and two.

(i.) Regarding $5x+6y=12$ and $3x-4y=30$ as simultaneous, and solving them, we find one vertex to be the point $(6, -3)$.

Solving $3x-4y=30$ and $x+5y=10$ as simultaneous equations, we find the second vertex to be $(10, 0)$.

Solving $x+5y=0$ and $5x+6y=12$ as simultaneous equations, we find the third vertex to be $(0, 2)$.

Hence the three vertices are $(6, -3)$, $(10, 0)$, and $(0, 2)$.

(ii.) First let (x_1, y_1) be $(6, -3)$, and (x_2, y_2) be $(10, 0)$; then, by [1], we have

$$d=\sqrt{4^2+3^2}=\sqrt{25}=5.$$

Now let (x_1, y_1) be $(10, 0)$ and (x_2, y_2) be $(0, 2)$; then, by [1], we have

$$d=\sqrt{(-10)^2+2^2}=\sqrt{104}=2\sqrt{26}.$$

Again, let (x_1, y_1) be $(0, 2)$, and (x_2, y_2) be $(6, -3)$; then, by [1], we have

$$d=\sqrt{6^2+(-5)^2}=\sqrt{61}.$$

Hence the lengths of the three sides are 5, $2\sqrt{26}$, and $\sqrt{61}$.

23. Find the lengths of the sides of a triangle if the equations of the sides are $x=0$, $y=0$, and $4x+3y=12$.

Since $y=0$ and $x=0$ are the axes of co-ordinates, the triangle is a right triangle formed by the locus of $4x+3y=12$ and its intercept on these axes. Now these intercepts are evidently 3 and 4 respectively; hence the sides about the right angle are 3 and 4. Hence, by Geometry, the third side is $\sqrt{3^2+4^2}$, or 5.

24. What are the vertices of the quadrilateral enclosed by the straight lines $x-a=0$, $x+a=0$, $y-b=0$, $y+b=0$? What kind of a quadrilateral is it?

The sides in order are evidently $x=a$, $y=b$, $x=-a$, and $y=-b$. The intersection of $x=a$ and $y=b$ is evidently (a, b); that of $y=b$ and $x=-a$ is $(-a, b)$; that of $x=-a$ and $y=-b$ is $(-a, -b)$; and that of $y=-b$ and $x=a$ is $(a, -b)$. Hence the vertices are (a, b), $(-a, b)$, $(-a, -b)$, $(a, -b)$.

Since $x=a$ and $x=-a$ are each parallel to the axis of y, and $y=b$ and $y=-b$ are each parallel to the axis of x, the enclosed figure is a rectangle.

25. Does the straight line $5x + 4y = 20$ cut the circle $x^2 + y^2 = 9$?

Regarding these equations as simultaneous, and solving, we obtain imaginary values for x and y; hence the plane loci of these equations do not intersect.

26. Find the length of that part of the straight line $3x - 4y = 0$ which is contained within the circle $x^2 + y^2 = 25$.

Regarding these equations as simultaneous, and solving for x and y, we obtain $x = 4$, $y = 3$, and $x = -4$, $y = -3$; hence the extremities of the chord are $(4, 3)$ and $(-4, -3)$.

Let (x_1, y_1) be $(4, 3)$, and (x_2, y_2) be $(-4, -3)$; then, by [1], we have $d = \sqrt{(-8)^2 + (-6)^2} = 10 =$ the length of the chord.

27. Which of the following curves pass through the origin of coordinates?

(1) $7x - 2y + 4 = 0$. (3) $y^2 - x^2 = 4y$. (5) $ax + by + c = 0$.
(2) $7x - 2y = 0$. (4) $ax + by = 0$. (6) $x^2 - y + a = a + xy$.

Since each of the equations (2), (3), and (4) has no constant term, and since (6) has none when simplified, the locus of each of these equations passes through the origin.

28. Change the equation $4x + 2y - 7 = 0$ so that its locus shall pass through the origin.

If we drop the constant term -7, we have $4x + 2y = 0$, the locus of which passes through the origin.

Ex. 8. Page 30.

1. Construct the locus of $3x - y - 2 = 0$.

Here $y = 3x - 2$.
When $x = 0, 1, 2, 3, 4, \ldots,$
$y = -2, 1, 4, 7, 10, \ldots.$

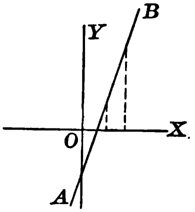

Fig. 11.

Locating the points $(0, -2)$, $(1, 1)$, $(2, 4)$, $(3, 7)$, ..., and tracing a continuous line through them, we obtain AB as the required locus.

That the locus is a straight line is evident from the fact that for all values of x, an increase of 1 in x causes an increase of 3 in y.

2. Construct the locus of $y = 2x$.

When $x = 0,\ 1,\ 2,\ 3,\ 4,\ \cdots\cdots,$

$y = 0,\ 2,\ 4,\ 6,\ 8,\ \cdots\cdots.$

Here y increases by 2 while x increases by 1; hence the locus is a straight line. Locating the point (1, 2) and drawing a straight line through it and the origin, we obtain AB (Fig. 12) as the required locus.

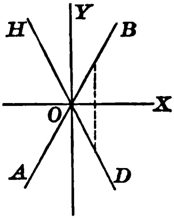

FIG. 12.

3. Construct the locus of $x^2 = y^2$.

Solving the given equation, we obtain

$$y = x, \qquad (1)$$

and

$$y = -x. \qquad (2)$$

The locus of (1) is evidently the straight line which bisects the first and third quadrants; while that of (2) is the straight line bisecting the second and fourth quadrants. Hence these two lines constitute the locus of the given equation.

4. Construct the locus of $x^2 + y^2 = 100$.

By [1], this equation states that the point (x, y) is at the distance 10 from the origin; hence the locus of the equation is the circle whose centre is the origin, and whose radius is 10.

5. Construct the locus of $x^2 - y^2 = 25$.

Here

$$y = \pm\sqrt{x^2 - 25}, \qquad (1)$$

and

$$x = \pm\sqrt{y^2 + 25}. \qquad (2)$$

In (1) for $x = 0$, y is imaginary; hence the locus does not cut the axis of y. In (2) for $y = 0$, $x = \pm 5$; hence the locus cuts the axis of x at 5 to the right and 5 to the left of the origin.

In (1) for any value of x between -5 and $+5$, y is imaginary; hence no part of the locus lies between the lines $x = -5$ and $x = +5$. Any positive value of x greater than $+5$ renders y real; hence the locus is continuous between $x = +5$ and $x = +\infty$. For like reason, the locus is continuous between $x = -5$ and $x = -\infty$. Any real value of y in (2) renders x real; hence the locus is continuous in the direction of the axis of y from $y = -\infty$ to $y = +\infty$.

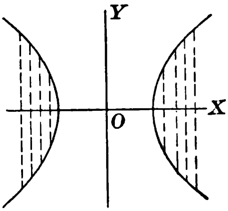

FIG. 13.

$$\text{When}\quad \left.\begin{array}{llllll} x = & 5, & 6, & 7, & 8, & 9, \cdots\cdots, \\ y = & 0, & \pm 3.3, & \pm 4.9, & \pm 6.2, & \pm 7.5, \cdots\cdots. \end{array}\right\} \qquad (3)$$

$$\text{When}\quad \left.\begin{array}{llllll} x = & -5, & -6, & -7, & -8, & -9, \cdots\cdots, \\ y = & 0, & \pm 3.3, & \pm 4.9, & \pm 6.2, & \pm 7.5, \cdots\cdots. \end{array}\right\} \qquad (4)$$

Locating the points given in (3) and (4), and tracing through them a continuous curve, we obtain the locus, which consists of the two branches given in Fig. 13.

Since the given equation contains only the second powers of x and y, if (x_1, y_1) be a point on the locus, $(x_1, -y_1)$ and $(-x_1, y_1)$ are each on the locus also. But $(x_1, -y_1)$ is the symmetrical of (x_1, y_1) with respect to the axis of x, and $(-x_1, y_1)$ is the symmetrical of (x_1, y_1) with respect to the axis of y. Hence the locus is symmetrical with respect to each of the co-ordinate axes.

6. Construct the locus of $4x^2 - y^2 = 0$.

The equation $4x^2 - y^2 = 0$ may be written in the form

$$(2x + y)(2x - y) = 0.$$

Hence the locus consists of the loci of the equations

$$y = 2x \quad\text{and}\quad y = -2x.$$

The locus of $y = 2x$ is the line AB (Fig. 12).

In $y = -2x$ we have,

$$\text{when}\quad \begin{array}{lllll} x = 0, & 1, & 2, & 3, & \cdots\cdots, \\ y = 0, & -2, & -4, & -6, & \cdots\cdots. \end{array}$$

Here y decreases by 2 while x increases by 1; hence the locus is a straight line. Locating the point $(1, -2)$ (Fig. 12), and drawing a straight line through it and the origin, we obtain HD as the locus of $y = -2x$. Hence the locus of $4x^2 - y^2 = 0$ consists of AB and HD.

7. Construct the locus of $4x^2 + 9y^2 = 144$.

$$\text{Here}\quad y = \pm \tfrac{2}{3}\sqrt{36 - x^2}.$$

$$\text{When}\quad \left.\begin{array}{llllllll} x = & 0, & 1, & 2, & 3, & 4, & 5, & 6, \\ y = & \pm 4, & \pm 3.9, & \pm 3.8, & \pm 3.5, & \pm 3, & \pm 2.2, & 0. \end{array}\right\} \quad (1)$$

$$\text{When}\quad \left.\begin{array}{lllllll} x = & -1, & -2, & -3, & -4, & -5, & -6, \\ y = & \pm 3.9, & \pm 3.8, & \pm 3.5, & \pm 3, & \pm 2.2, & 0. \end{array}\right\} \qquad (2)$$

When $x > +6$ or < -6, y is imaginary; hence no part of the locus lies to the right of the line $x = 6$, or to the left of the line $x = -6$.

Locating the points given in (1) and (2), and tracing through them a continuous curve, we obtain the curve in Fig. 14 as the locus of the given equation.

Since the given equation contains only the second powers of x and y, the locus is symmetrical with respect to each of the axes, and also with respect to the origin O.

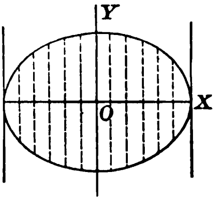

FIG. 14.

8. Construct the locus of $y^2 - 16x = 0$.

Here $y = \pm 4\sqrt{x}$.

For negative values of x, y is imaginary; hence the locus does not extend to the left of the axis of y. For every positive value of x, y has two real values numerically equal, with unlike signs; hence the locus consists of two infinite branches, one in the first, and the other in the fourth, quadrant, which are symmetrical with respect to the axis of x.

For $x = 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots\ldots,$

$y = 0, \quad \pm 4, \quad \pm 5.6, \quad \pm 6.9, \quad \pm 8, \quad \pm 8.9, \quad \ldots\ldots.$

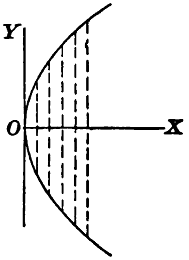

FIG. 15.

Locating the points given above, and tracing through them a continuous curve, we obtain the curve in Fig. 15 as the locus of $y^2 = 16x$.

9. Construct the locus of $y^2 + 16x = 0$.

Here $y = \pm 4\sqrt{-x}$.

For positive values of x, y is imaginary; hence the locus does not extend to the right of the axis of y. For every negative value of x, y has two real values numerically equal, with opposite signs; hence the locus consists of two infinite branches, one in the second, and the other in the third, quadrant, which are symmetrical with respect to the axis of x. When

$x = 0, \quad -1, \quad -2, \quad -3, \quad -4, \quad -5, \quad \ldots\ldots,$

$y = 0, \quad \pm 4, \quad \pm 5.6, \quad \pm 6.9, \quad \pm 8, \quad \pm 8.9, \quad \ldots\ldots.$

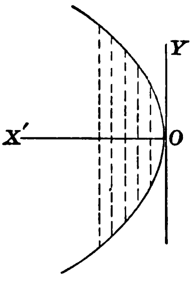

FIG. 16.

Locating the points given above, and tracing through them a continuous curve, we obtain the curve in Fig. 16 as the locus of $y^2 + 16x = 0$.

The curves in Figs. 15 and 16 evidently differ only in position.

10. Construct the locus of $x^2 - 2x - 10y - 5 = 0$.

Solving the equation for x, we obtain

$$x = 1 \pm \sqrt{10y + 6}.$$

For $y = -0.6$, $x = 1$, and for $y < -0.6$, x is imaginary; hence no part of the locus lies below the line $y = -0.6$. For any value of y greater than -0.6, x has two real values; hence the locus consists of two infinite branches, one in the first, and the other in the second, quadrant. When

$y =$	-0.6	0	1	2	3	4	…..,
$x =$	1	$+3.4, -1.4$	$5, -3$	$6.1, -4.1$	$7, -5$	$7.8, -5.8$	…… .

Locating the points given above, and tracing through them a continuous curve, we obtain the curve in Fig. 17 as the locus of

$$x^2 - 2x - 10y - 5 = 0.$$

This curve is evidently symmetrical with respect to the line $x = 1$, and its branches are infinite.

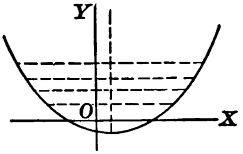

FIG. 17.

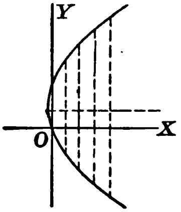

FIG. 18.

11. Construct the locus of $y^2 - 2y - 10x = 0$.

Here $$y = 1 \pm \sqrt{10x + 1}.$$

For $x = -0.1$, $y = 1$, and for $x < -0.1$, y is imaginary; hence no part of the locus lies to the left of the line $x = -0.1$. For any value of x greater than 0.1, y has two real values; hence the locus consists of two infinite branches, one in the first, and the other in the fourth, quadrant. When

$x =$	-0.1	0	1	2	3	4	…..,
$y =$	1	$2, 0$	$4.3, -2.3$	$5.6, -3.6$	$6.6, -4.6$	$7.4, -5.4$	…… .

Locating the points given by the values of x and y above, and tracing through them a continuous curve, we obtain the curve in Fig. 18 as the locus of $y^2 - 2y - 10x = 0$.

This curve is evidently symmetrical with respect to the line $y = 1$.

12. Construct the locus of $(x-3)^2+(y-2)^2=25$.

By [1] this equation states that the point (x, y) is at the distance 5 from the point $(3, 2)$; hence the locus of this equation is the circle whose centre is the point $(3, 2)$, and whose radius is 5.

13. Construct the locus of $y^2-1=0$.

This equation may be written in the form

$$(y+1)(y-1)=0;$$

hence its locus consists of the two lines $y=1$ and $y=-1$; that is, the locus consists of two lines parallel to the axis of x, one 1 above that axis, and the other 1 below it.

14. Construct the locus of $y=x^3$.

Here for any real value of x, y has one real value, and has the same sign as x. Hence the locus has one infinite branch in the first quadrant, and another in the third.

$$\text{When} \quad \left.\begin{array}{l} x = \;\; 0, \quad 1, \quad 2, \quad 3, \quad 4, \;\cdots\cdots, \\ y = \;\; 0, \quad 1, \quad 8, \quad 27, \quad 64, \;\cdots\cdots. \end{array}\right\} \qquad (1)$$

$$\text{When} \quad \left.\begin{array}{l} x = -1, \; -2, \; -3, \; -4, \;\cdots\cdots, \\ y = -1, \; -8, \; -27, \; -64, \;\cdots\cdots. \end{array}\right\} \qquad (2)$$

Locating the points given in (1) and (2), and tracing through them a continuous curve, we obtain the curve in Fig. 19 as the locus of $y=x^3$. Since, if (x_1, y_1) is a point on the curve, $(-x_1, -y_1)$ is also on it; the locus is symmetrical with respect to the origin as a centre of symmetry.

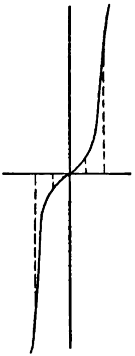

FIG. 19.

15. Construct the locus of $xy=12$.

Here $\quad y=\dfrac{12}{x}$ and $x=\dfrac{12}{y}$.

Since their product is positive, x and y must have like signs; hence the locus lies in the first and third quadrants.

$$\text{When} \quad \begin{array}{l} x = \;\; 1, \;\; 2, \;\; 3, \;\; 4, \;\; 6, \;\; 12, \;\cdots\cdots, \\ y = 12, \;\; 6, \;\; 4, \;\; 3, \;\; 2, \;\; 1, \;\cdots\cdots. \end{array}$$

Locating the points given above, and noting that as x approaches infinity from 12, y approaches zero; and that as x approaches zero from 1, y approaches infinity, we may trace the part of the locus that lies in the first quadrant, as in Fig. 20. Since, if (x_1, y_1) is a point on $xy = 12$, $(-x_1, -y_1)$ is also a point on it; the part of the locus in the third quadrant may be obtained by drawing the symmetrical of the branch in the first quadrant with respect to the origin O. Thus, to construct the symmetrical of P, draw PO, and on it produced lay off OP' equal to OP; then is P' one point on the branch in the third quadrant. In like manner, any number of other points may be located.

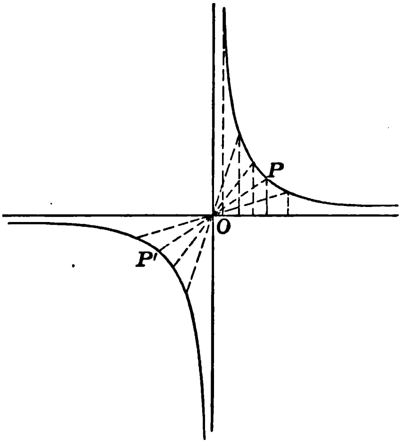

FIG. 20.

16. Construct the locus of $x = \sin y$.

When $y = 0° = 0$, $x = 0$. When $y = 50° = 0.87$, $x = 0.77$.
$y = 10° = 0.17$, $x = 0.17$. $y = 60° = 1.05$, $x = 0.87$.
$y = 20° = 0.35$, $x = 0.34$. $y = 70° = 1.22$, $x = 0.94$.
$y = 30° = 0.52$, $x = 0.50$. $y = 80° = 1.40$, $x = 0.98$.
$y = 40° = 0.70$, $x = 0.64$. $y = 90° = 1.57$, $x = 1.00$.

If we continue the values of y from 90° to 180°, the above values of x repeat themselves in the inverse order; from 180° to 360° the values of x are numerically the same, and occur in the same order as between 0° and 180°, but are negative.

Locating the points given above, and tracing a continuous curve through them, we obtain the locus from O to M in Fig. 21. It is evident that this portion of the curve will repeat itself at intervals of 2π, both above and below the origin.

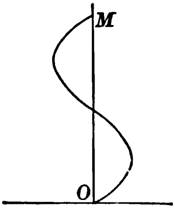

FIG. 21.

17. Construct the locus of $y = 2 \sin x$.

This and the following loci may be constructed with sufficient accuracy as follows:

Divide the quadrant AK of the unit circle into eight equal parts; then, without much error, the chords of these partial arcs may be taken as the arcs themselves.

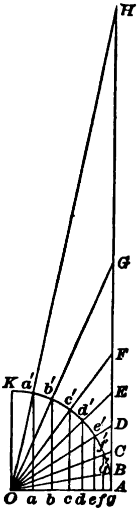

FIG. 22.

To construct $y = 2 \sin x$, lay off in succession, on the axis of x (Fig. 23), the distances 0 1, 1 2, 2 3, 3 4, 4 5, 5 6, etc., equal to the chord Ag' (Fig. 22). On the perpendiculars at 1, 2, 3, 4, 5, etc., lay off $1m = 2gg'$, $2n = 2ff'$, $3p = 2ee'$, $4q = 2dd'$, $5r = 2cc'$, $6s = 2bb'$, $7t = 2aa'$, $8u = 2OK$, $9v = 2aa'$, $10w = 2bb'$, etc., and trace a continuous curve through the points O, m, n, p, etc.

It is evident that the values of $2 \sin x$ from 90° to 180° are those of $2 \sin x$ from 0° to 90° in reverse order. Those of $2 \sin x$ from 180° to 360° are those of $2 \sin x$ from 0° to 180° with the minus sign prefixed, so that the branch below the axis of x will be of the same form as that given in the figure.

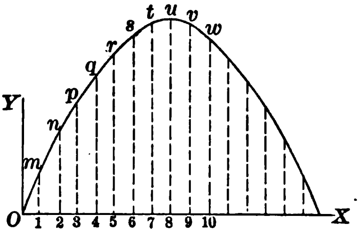

FIG. 23.

It is evident that the curve will repeat itself at intervals of 2π, both to the right and to the left of the origin.

18. Construct the locus of

$$y = \sin 2x.$$

On OX (Fig. 24) lay off the distances 0 1, 1 2, etc., equal to chord Ag' (Fig. 22) as above.

On the perpendiculars at these points lay off $1m = ff'$, $2n = dd'$, $3p = bb'$, $4q = OK$, $5r = bb'$, $6s = dd'$, $7t = ff'$, etc. Through the points thus located trace a continuous curve. The curve cuts the axis

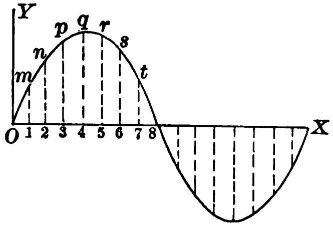

FIG. 24.

of x at intervals of $\frac{1}{2}\pi$, extends both to the right and the left of the origin, and repeats itself at intervals of π.

19. Construct the locus of $y = \cos x$.

On OX (Fig. 25) lay off the distances equal to Ag' (Fig. 22) as above.

On the perpendiculars at the points thus determined lay off $OK = OA$, $1m = Og$, $2n = Of$, $3p = Oe$, $4q = Od$, $5r = Oc$, etc.

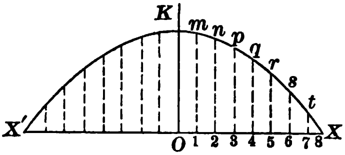

FIG. 25.

Since $\cos(-x) = \cos x$, lay off these distances in the same order on the perpendiculars to the left of the origin, and through the points thus determined trace a continuous curve. The curve cuts the axis of x at $\frac{1}{2}\pi$ to the right and to the left of the origin, and in general at intervals of π. To the right of 8 there is a portion below the axis of x of the same shape and dimensions as that given in Fig. 25, and the curve repeats itself at intervals of 2π.

20. Construct the locus of $y = \tan x$.

On OX and OX' (Fig. 26) lay off the distances equal to Ag' (Fig. 22), and on the perpendiculars at the points thus determined lay off, successively, distances equal to $AB, AC, AD, AE, AF, AG, AH$; those to the right of the origin being laid off upward, and those to the left downward.

Through the points thus determined trace a continuous curve.

As x approaches $\frac{1}{2}\pi$, $\tan x$ approaches ∞, and as x approaches $-\frac{1}{2}\pi$, $\tan x$ approaches $-\infty$; hence the infinite branch in the first quadrant approaches indefinitely near the line $x = \frac{1}{2}\pi$, and that in the third approaches the line $x = -\frac{1}{2}\pi$. The curve cuts the axis of x at the origin and at intervals of π to the right and the left of the origin. The curve repeats itself at intervals of π, both to the right and the left, one entire branch lying between the lines $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$.

21. Construct the locus of $y = \cot x$.

On OX (Fig. 27) lay off the equal distances as before, and erect the perpendiculars. Since $\cot x = \tan(\frac{1}{2}\pi - x)$, on these perpendiculars lay off upward distances equal to $AH, AG, AF, AE, AD, AC, AB$, and then in reverse order lay them off downward. Trace one branch of the curve through the points thus determined. As x approaches zero, $\cot x$ approaches ∞, and as x approaches π, $\cot x$ approaches $-\infty$; hence the

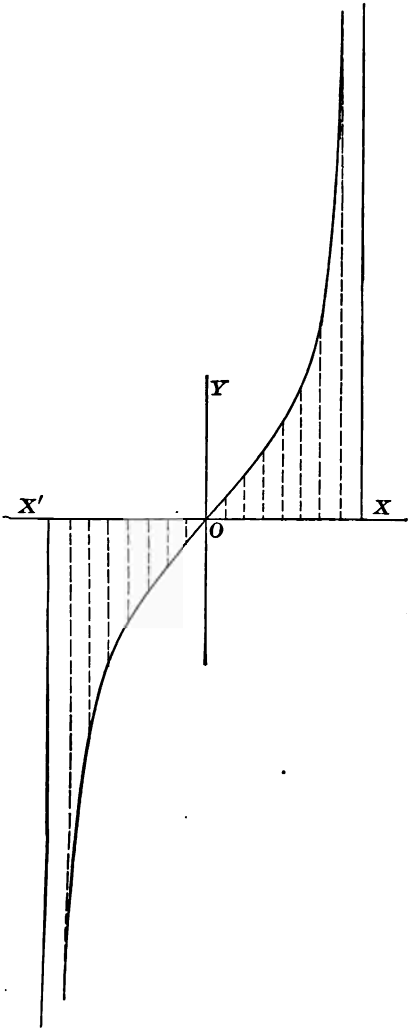

Fig. 26.

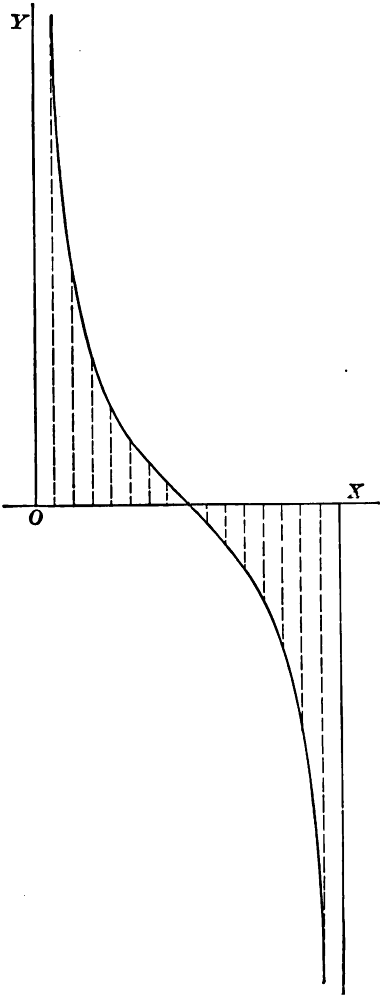

Fig. 27.

infinite arms of this branch approach infinitely near the axis of y and the line $x = \pi$. The curve cuts the axis of x at $\frac{1}{2}\pi$ to the right of the origin, and at intervals of π, both to the right and to the left of this point. The curve repeats itself at intervals of π, both to the right and left of the origin, one entire branch lying between the axis of y and the line $x = \pi$.

22. Construct the locus of $y = \sec x$.

On OX and OX' lay off the distances equal to Ag' (Fig. 22), and on the perpendiculars at the points thus determined lay off successively distances equal to OB, OC, OD, OE, OF, OG, OH, upward on both sides of the origin, as $\sec x$ is $+$ for all values of x between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Through the points thus determined (Fig. 28) draw the curve which cuts the axis of y at $(0, 1)$.

The infinite arms of the branch in the figure approach indefinitely near the lines $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$. The curve does not cut the axis of x. The curve repeats itself at intervals of π, both to the right and the left of $x = \frac{1}{2}\pi$, the branches being alternately above and below the axis of x.

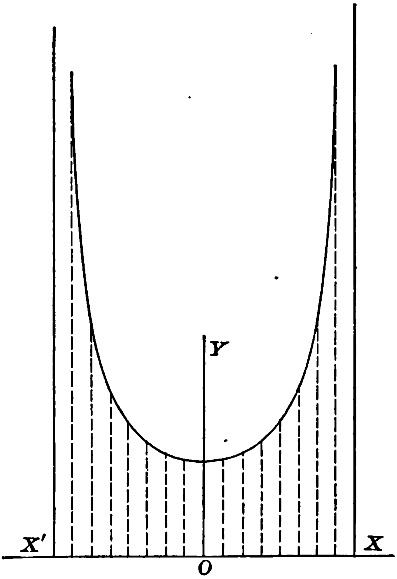

FIG. 28.

23. Construct the locus of $y = \csc x$.

On OX (Fig. 29) lay off the equal distances as before, and erect the perpendiculars. Since $\csc x = \sec(\frac{1}{2}\pi - x)$, on these perpendiculars lay off upward distances equal to OH, OG, ..., OB, OA, OB, ..., OH, and trace a continuous curve through the points thus determined. As x approaches zero, $\csc x$ approaches ∞, and as x approaches π, $\csc x$ approaches $+\infty$; hence the infinite arms of this branch approach indefinitely near the axis of y and the line $x = \pi$. The curve repeats itself at intervals of π, both to the right and the left of the origin, the branches being alternately above and below the axis of x.

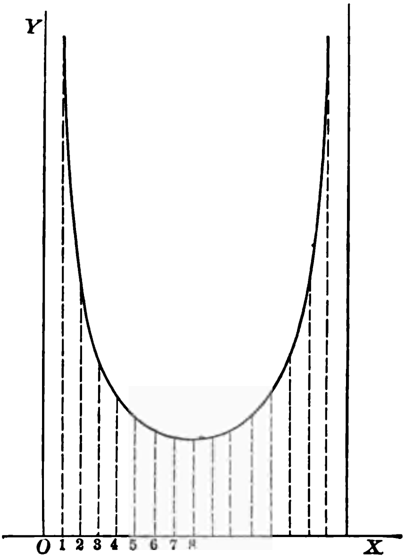

FIG. 29.

24. Construct the locus of $y = \sin x + \cos x$.

On OX and OX' (Fig. 30) lay off the equal distances as before, and erect perpendiculars. On the perpendiculars to the right of O lay off distances equal to $gg' + Og$, $ff' + Of$, $ee' + Oe$, ..., $aa' + Oa$, OK, $aa' - Oa$, $bb' - Ob$, $dd' - Od$. On the perpendiculars to the left of O lay off distances equal to $Og - gg'$, $Of - ff'$, $Oe - ee'$, $Od - dd'$. We thus determine the points corresponding to the values of x between $-\frac{1}{4}\pi$ and $\frac{3}{4}\pi$. Drawing a continuous curve through these points, we obtain the branch in Fig. 30. The portions, or branches, below the axis of x are of the same form and dimensions as the branch in the figure, since the negative values of $\sin x + \cos x$ for $x > \frac{3}{4}\pi$ and $< \frac{7}{4}\pi$ are numerically equal to the positive values of $\sin x + \cos x$ for $x > -\frac{1}{4}\pi$ and $< \frac{3}{4}\pi$. For $y = 0$, we have $\sin x = -\cos x$ or $\tan x = -1$; hence the curve cuts the axis of x at $\frac{1}{4}\pi$ to the left of O, and $\frac{3}{4}\pi$ to the right of O, and at intervals of π both to the right and to the left of these points. The curve evidently repeats itself at intervals of 2π both to the right and to the left.

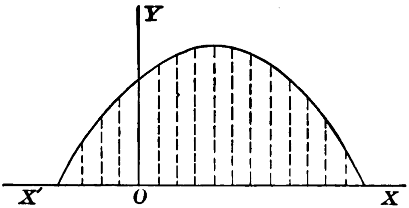

FIG. 30.

Ex. 9. Page 31.

1. A point moves so that it is always three times as far from the axis of x as from the axis of y. What is the equation of its locus?

From the conditions of the problem, y, or the distance of the point from the axis of x, equals $3x$, or three times the distance of the point from the axis of y; hence $y = 3x$ is the equation of the locus of the point.

2. What is the equation of the locus of a point which moves so that its abscissa is always equal to $+6$? -6? 0?

In the first case the point must trace a straight line parallel to the axis of y at the distance 6 to the right of it, the equation of which is $x = 6$. For like reason the equation in the second case is $x = -6$; and in the third $x = 0$.

3. What is the equation of the locus of a point which moves so that its ordinate is always equal to $+4$? -1? 0?

In the first case the point must trace a straight line parallel to the axis of x at the distance 4 above it, the equation of which is $y = 4$.

For like reason the equation in the second case is $y = -1$; and in the third $y = 0$.

4. A point so moves that its distance from the straight line $x = 3$ is always numerically equal to 2. What is the equation of its locus?

From the conditions, the point must trace a straight line parallel to the line $x = 3$, at the distance 2 to the right, or 2 to the left, of it; hence the equations of the locus are $x = 5$, and $x = 1$, or $x - 5 = 0$ and $x - 1 = 0$, which may be written

$$(x - 5)(x - 1) = x^2 - 6x + 5 = 0.$$

5. A point so moves that its distance from the straight line $y = 5$ is always numerically equal to 3. Find the equation of its locus. Construct the locus.

The point must trace a line parallel to the line $y = 5$ at the distance 3 above or 3 below it; hence the equations of the locus are $y = 8$ and $y = 2$, which may be written

$$(y - 8)(y - 2) = y^2 - 10y + 16 = 0.$$

6. A point moves so that its distance from the straight line $x+4=0$ is always numerically equal to 5. Find the equation of its locus. Construct the locus.

The point must trace a line parallel to the line $x=-4$ at the distance 5 to the right or to the left of it; hence the equations of the locus are $x=1$ and $x=-9$, which may be written

$$(x-1)(x+9)=x^2+8x-9=0.$$

7. What is the equation of the locus of a point equidistant

(1) from the parallels $x=0$ and $x=-6$?

(2) from the parallels $y=7$ and $y=-3$

(1) The parallels are the axis of y and a line 6 to the left of it; hence the point must be on a line 3 to the left of the axis of y, whose equation is $x=-3$, or $x+3=0$.

(2) The parallels are parallel to the axis of x, one 7 above it, and the other 3 below; hence the point midway between them must be 2 above that axis, and the equation of the locus is $y=2$, or $y-2=0$.

8. What is the equation of the locus of a point always equidistant from the origin and the point $(6,0)$?

Let (x,y) be the point; then, by [1], its distance from the origin $(0,0)$ is $\sqrt{x^2+y^2}$, and its distance from $(6,0)$ is $\sqrt{(x-6)^2+y^2}$. Putting these distances equal to each other, we have

$$\sqrt{x^2+y^2}=\sqrt{(x-6)^2+y^2},$$

or $$x=3,$$

which is the equation required.

9. Find the equation of the locus of a point equidistant from the points $(4,0)$ and $(-2,0)$.

Let (x,y) be the point; then, by [1], its distance from $(4,0)$ is $\sqrt{(x-4)^2+y^2}$, and its distance from $(-2,0)$ is $\sqrt{(x+2)^2+y^2}$. Putting these distances equal to each other, we have

$$\sqrt{(x-4)^2+y^2}=\sqrt{(x+2)^2+y^2},$$

or $$x=1,$$

which is the equation required.

10. Find the equation of the locus of a point equidistant from the points $(0, -5)$ and $(0, 9)$.

Let (x, y) be the point; then, by [1], its distance from $(0, -5)$ is $\sqrt{x^2 + (y + 5)^2}$, and its distance from $(0, 9)$ is $\sqrt{x^2 + (y - 9)^2}$. Putting these distances equal to each other, we have

$$\sqrt{x^2 + (y + 5)^2} = \sqrt{x^2 + (y - 9)^2},$$

or

$$y = 2,$$

which is the equation required.

11. Find the equation of the locus of a point equidistant from the points $(3, 4)$ and $(5, -2)$.

Let (x, y) be the point; then, by [1], its distance from $(3, 4)$ is $\sqrt{(x-3)^2 + (y-4)^2}$, and its distance from $(5, -2)$ is $\sqrt{(x-5)^2 + (y+2)^2}$. Putting these distances equal to each other, we have

$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-5)^2 + (y+2)^2},$$

or

$$y = \tfrac{1}{3}x - \tfrac{1}{3},$$

which is the equation required.

12. Find the equation of the locus of a point equidistant from the points $(5, 0)$ and $(0, 5)$.

Let (x, y) be the point; then, by [1], its distance from $(5, 0)$ is $\sqrt{(x - 5)^2 + y^2}$, and its distance from $(0, 5)$ is $\sqrt{x^2 + (y - 5)^2}$. Putting these distances equal to each other, we have

$$\sqrt{(x - 5)^2 + y^2} = \sqrt{x^2 + (y - 5)^2},$$

or

$$y = x,$$

which is the equation required.

13. A point moves so that its distance from the origin is always equal to 10. Find the equation of its locus.

Let (x, y) be the point; then, by [1], its distance from the origin is $\sqrt{x^2 + y^2}$. Putting this equal to 10, we have

$$\sqrt{x^2 + y^2} = 10,$$

or

$$x^2 + y^2 = 100,$$

as the required equation.

14. A point moves so that its distance from the point $(4, -3)$ is always equal to 5. Find the equation of its locus, and construct it. What kind of curve is it? Does it pass through the origin? Why?

Let (x, y) be the point; then, by [1], its distance from the point $(4, -3)$ is $\sqrt{(x-4)^2 + (y+3)^2}$. Putting this equal to 5, we obtain

$$\sqrt{(x-4)^2 + (y+3)^2} = 5,$$

or
$$(x-4)^2 + (y+3)^2 = 25, \qquad (1)$$

or
$$x^2 + y^2 - 8x + 6y = 0, \qquad (2)$$

as the required equation.

Since the moving point is at the distance 5 from the point $(4, -3)$, the locus is a circle whose centre is $(4, -3)$ and whose radius is 5.

The locus passes through the origin, since in the simplest form its equation (2) has no constant term.

15. What is the equation of the locus of a point whose distance from the point $(-4, -7)$ is always equal to 8?

Let (x, y) be the point; then, by [1], its distance from the point $(-4, -7)$ is $\sqrt{(x+4)^2 + (y+7)^2}$. Putting this equal to 8, we obtain

$$\sqrt{(x+4)^2 + (y+7)^2} = 8,$$

or
$$(x+4)^2 + (y+7)^2 = 64,$$

as the required equation.

16. About the origin of co-ordinates as centre, with a radius equal to 5, a circle is described. A point outside this circle so moves that its distance from the circumference of the circle is always equal to 4. What is the equation of its locus?

Since the point is always without, and at the distance 4 from the circumference whose radius is 5, it must be at the distance 9 from the origin.

Let (x, y) be the point; then, from [1], its distance from the origin is $\sqrt{x^2 + y^2}$. Putting this equal to 9, we obtain

$$\sqrt{x^2 + y^2} = 9,$$

or
$$x^2 + y^2 = 81,$$

as the required equation.

17. A high rock A, rising out of the water, is 3 miles from a perfectly straight shore BC. A vessel so moves that its distance from the rock is always the same as its distance from the shore. What is the equation of its locus?

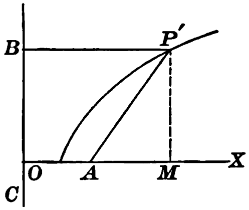

FIG. 31.

Let A represent the position of the high rock, and CB that of the straight shore.

Through A draw XO perpendicular to CB; then $OA = 3$. Take OX as the axis of x, and OB as the axis of y. Let $P'(x, y)$ denote any position of the vessel. Draw $P'M$ parallel to BC; then

$$AP'^2 = AM^2 + MP'^2. \qquad (1)$$

But
$$AP' = BP' = OM = x, \quad MP' = y.$$
$$AM = OM - OA = x - 3.$$

Substituting these values in (1), we obtain

$$x^2 = (x-3)^2 + y^2,$$

or
$$y^2 - 6x + 9 = 0,$$

as the required equation.

18. A point A is situated at the distance 6 from the line BC. A moving point P is always equidistant from A and BC. Find the equation of its locus.

Let A (Fig. 31) be the fixed point, and BC the fixed straight line. Through A draw XO perpendicular to CB; then $OA = 6$. Take OX as the axis of x, and OB the axis of y. Let $P'(x, y)$ be any position of the moving point P. Draw $P'M$ parallel to BC; then

$$AP'^2 = AM^2 + MP'^2. \qquad (1)$$

But
$$AP' = BP' = OM = x, \quad MP' = y,$$
$$AM = OM - OA = x - 6.$$

Substituting these values in (1), we obtain

$$x^2 = (x-6)^2 + y^2,$$

or
$$y^2 - 12x + 36 = 0,$$

as the required equation.

19. A point moves so that its distance from the axis of x is half its distance from the origin; find the equation of its locus.

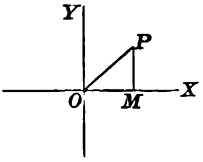

FIG. 32.

Let OX and OY be the axes, and P any position of the moving point (x, y). Draw PM parallel to OY, and draw OP; then in the right triangle OPM, we have

$$OP^2 = OM^2 + MP^2. \quad (1)$$

But $\quad OM = x, \quad MP = y,$

and since P is twice as far from O as from XO,

$$OP = 2\,MP = 2\,y.$$

Substituting these values in (1), we have

$$(2\,y)^2 = x^2 + y^2,$$

or

$$3\,y^2 = x^2,$$

as the required equation.

20. A point moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is the constant $2\,k^2$; find the equation of its locus.

Let (x, y) be the moving point; then, by [1], its distance from $(a, 0)$ is $\sqrt{(x-a)^2 + y^2}$, and its distance from $(-a, 0)$ is $\sqrt{(x+a)^2 + y^2}$. Putting the sum of the squares of these distances equal to $2\,k^2$, we obtain

$$(x-a)^2 + y^2 + (x+a)^2 + y^2 = 2\,k^2,$$

or

$$x^2 + y^2 = k^2 - a^2,$$

as the required equation.

21. A point moves so that the difference of the squares of its distances from $(a, 0)$ and $(-a, 0)$ is the constant k^2; find the equation of its locus.

Let (x, y) be the moving point; then, by [1], its distance from $(a, 0)$ is $\sqrt{(x-a)^2 + y^2}$, and its distance from $(-a, 0)$ is $\sqrt{(x+a)^2 + y^2}$. Putting the difference of the squares of these distances equal to k^2, we obtain

$$(x-a)^2 + y^2 - (x+a)^2 - y^2 = \pm k^2,$$

or

$$4\,ax = \pm k^2,$$

as the required equation.

Ex. 10. Page 33.

1. If we should plot all possible points for which $x = -5$, how would they be situated?

Each point would be at the distance 5 to the left of the axis of y, and hence in a line parallel to that axis, 5 to its left.

2. Construct the point (x, y) if $x = 2$ and

$$(1)\ y = 4x - 3, \quad (2)\ 3x - 2y = 8.$$

If in $y = 4x - 3$ we put $x = 2$, we obtain

$$y = 8 - 3 = 5;$$

hence in the first case, (x, y) is $(2, 5)$.

If in $3x - 2y = 8$ we put $x = 2$, we obtain

$$6 - 2y = 8, \text{ or } y = -1;$$

hence in the second case, (x, y) is $(2, -1)$.

3. The vertices of a rectangle are the points (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$. Find the lengths of its sides, the lengths of its diagonals, and show that the vertices are equidistant from the origin.

Since the ordinates of the vertices (a, b) and $(-a, b)$ are each b, the side between these vertices is parallel to the axis of x at the distance b above it, and the length of this side is $a - (-a)$, or $2a$. For like reason, the side between the vertices $(-a, -b)$ and $(a, -b)$ is parallel to the axis of x at the distance b below it, and the length of the side is $2a$. Since the abscissas of the vertices $(-a, b)$ and $(-a, -b)$ are each $-a$, the side between them is parallel to the axis of y at the distance a to the left, and the length of the side is $b - (-b)$, or $2b$. For like reason, the side between the vertices $(a, -b)$ and (a, b) is parallel to the axis of y at the distance a to the right, and the length of the side is $2b$. The opposite sides being parallel to the rectangular axes, the figure is a rectangle whose opposite sides are $2a$ and $2b$, respectively.

Now, by [1], the length of the diagonal between the vertices (a, b) and $(-a, -b)$ is $\sqrt{(a + a)^2 + (b + b)^2} = 2\sqrt{a^2 + b^2}$.

We obtain the same expression for the distance between the vertices $(-a, b)$ and $(a, -b)$.

Moreover, by [1], the distance from $(0, 0)$ to either of the vertices is $\sqrt{a^2 + b^2}$.

4. What does equation [1], p. 6, for the distance between two points, become when one of the points is the origin?

Let (x_2, y_2) be $(0, 0)$; then [1] becomes

$$d = \sqrt{x_1^2 + y_1^2}.$$

That is, *the distance of any point from the origin is equal to the square root of the sum of the squares of its rectangular co-ordinates.*

5. Express by an equation that the distance of the point (x, y) from the point $(4, 6)$ is equal to 8.

By [1], the distance from (x, y) to $(4, 6)$ is $\sqrt{(x-4)^2 + (y-6)^2}$. Putting this equal to 8, we obtain

$$\sqrt{(x-4)^2 + (y-6)^2} = 8,$$

or
$$x^2 + y^2 - 8x - 12y = 12,$$

as the required equation.

6. Express that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$.

By [1], the distance from (x, y) to $(2, 3)$ is $\sqrt{(x-2)^2 + (y-3)^2}$, and the distance from (x, y) to $(4, 5)$ is $\sqrt{(x-4)^2 + (y-5)^2}$.

Equating these two distances, we obtain

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-4)^2 + (y-5)^2},$$

or
$$x + y = 7,$$

as the required equation.

7. Find the point equidistant from the points $(2, 3)$, $(4, 5)$, and $(6, 1)$. What is the common distance?

By No. 6, if (x, y) is equidistant from $(2, 3)$ and $(4, 5)$, we have

$$x + y = 7. \tag{1}$$

Similarly, if (x, y) is equidistant from $(2, 3)$ and $(6, 1)$, we have

$$(x-2)^2 + (y-3)^2 = (x-6)^2 + (y-1)^2,$$

or
$$2x - y = 6. \tag{2}$$

Solving (1) and (2) as simultaneous, we find $x = \frac{13}{3}$ and $y = \frac{8}{3}$; hence the required point is $(\frac{13}{3}, \frac{8}{3})$.

By [1], the distance from $(6, 1)$ to $(\frac{13}{3}, \frac{8}{3})$ is

$$\sqrt{(6-\tfrac{13}{3})^2 + (1-\tfrac{8}{3})^2}, \text{ or } \tfrac{5}{3}\sqrt{2}.$$

8. Prove that the diagonals of a rectangle are equal.

Take two sides as axes, and denote their lengths by a and b; then the opposite vertices are $(0, 0)$ and (a, b), and $(a, 0)$ and $(0, b)$. The distance from $(0, 0)$ to (a, b) is $\sqrt{a^2 + b^2}$; and the distance from $(a, 0)$ to $(0, b)$ is also $\sqrt{a^2 + b^2}$; hence the diagonals are equal.

9. Prove that the diagonals of a parallelogram mutually bisect each other.

Let $OABC$ be any parallelogram; take OA as the axis of x, and OY as the axis of y. Let $OA = a$, $OH = AK = b$, $HC = c$; then O is $(0, 0)$, A is $(a, 0)$, B is $(a + b, c)$, and C is (b, c).

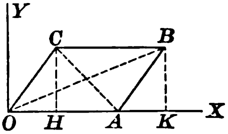

FIG. 33.

The mid-point between $O(0, 0)$ and $B(a + b, c)$ is $\left(\frac{a + b}{2}, \frac{c}{2}\right)$; and the mid-point between $A(a, 0)$ and $C(b, c)$ is $\left(\frac{a + b}{2}, \frac{c}{2}\right)$. Hence the middle points of the lines OB and AC coincide; that is, the diagonals mutually bisect each other.

10. The co-ordinates of three vertices of a parallelogram are $(5, 3)$, $(7, 10)$, $(13, 9)$. What are the co-ordinates of the remaining vertex?

(i.) If $(5, 3)$ and $(7, 10)$ are opposite vertices, the mid-point of the diagonals is, by [2], $(6, \frac{13}{2})$.

Now, if in [2] the mid-point (x, y) is $(6, \frac{13}{2})$, and (x_1, y_1) is $(13, 9)$, we have

$$6 = \frac{13 + x_2}{2}, \text{ or } x_2 = -1,$$

and

$$\frac{13}{2} = \frac{9 + y_2}{2}, \text{ or } y_2 = 4.$$

Hence the other vertex is $(-1, 4)$.

(ii.) If $(5, 3)$ and $(13, 9)$ are opposite vertices, the mid-point of the diagonals is $(9, 6)$.

Now, if in [2] the mid-point (x, y) be $(9, 6)$, and (x_1, y_1) be $(7, 10)$, we have

$$9 = \frac{7 + x_2}{2}, \text{ or } x_2 = 11,$$

and
$$6 = \frac{10 + y_2}{2}, \text{ or } y_2 = 2.$$

Hence the other vertex is (11, 2).

(iii.) If (7, 10) and (13, 9) are opposite vertices, the mid-point of the diagonals is $(10, \frac{19}{2})$.

Here the mid-point (x, y) is $(10, \frac{19}{2})$, and (x_1, y_1) is (5, 3); hence

$$10 = \frac{5 + x_2}{2}, \text{ or } x_2 = 15,$$

and
$$\frac{19}{2} = \frac{3 + y_2}{2}, \text{ or } y_2 = 16.$$

Hence the other vertex is (15, 16).

11. The co-ordinates of the vertices of a triangle are (3, 5) (7, −9), (2, −4). Find the co-ordinates of the middle points of its sides.

The mid-point between (3, 5) and (7, −9) is $\left(\frac{3+7}{2}, \frac{5-9}{2}\right)$ or (5, −2). The mid-point between (7, −9) and (2, −4) is $(\frac{9}{2}, -\frac{13}{2})$. The mid-point between (2, −4) and (3, 5) is $(\frac{5}{2}, \frac{1}{2})$.

12. The centre of gravity of a triangle is situated on the line joining any vertex to the middle point of the opposite side, at the point of trisection nearest that side. Find the centre of gravity of the triangle whose vertices are the points (2, 3), (4, −5), (−3, −6).

The mid-point between the vertices (2, 3) and (4, −5) is (3, −1).

If, in [3], (x_1, y_1) is (3, −1), (x_2, y_2) is (−3, −6), and $m : n$ is 1 : 2, we have

$$x = \frac{-3 + 6}{3} = 1, \quad y = \frac{-6 - 2}{3} = -\frac{8}{3}.$$

Hence the centre of gravity of the triangle is $(1, -\frac{8}{3})$.

13. The vertices of a triangle are (5, −3), (7, 9), (−9, 6). Find the distance from its centre of gravity to the origin.

The mid-point between (5, −3) and (7, 9) is (6, 3). Hence in [3], (x_1, y_1) is (6, 3), (x_2, y_2) is (−9, 6), and $m : n$ is 1 : 2, and we have

$$x = \frac{-9 + 12}{3} = 1, \quad y = \frac{6 + 6}{3} = 4.$$

Hence the centre of gravity is (1, 4), whose distance from the origin is $\sqrt{1 + 16}$ or $\sqrt{17}$.

14. The vertices of a quadrilateral are $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$. Find the co-ordinates of the intersection of the two straight lines which join the middle points of its opposite sides.

The vertices in order are $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$. The mid-point of the first side is $(\frac{5}{2}, 0)$; that of the second is $(7, \frac{11}{2})$; that of the third is $(\frac{9}{2}, 7)$; and that of the fourth is $(0, \frac{3}{2})$.

The line through $(\frac{5}{2}, 0)$ and $(\frac{9}{2}, 7)$ is

$$\frac{y}{x-\frac{5}{2}} = \frac{7}{\frac{9}{2}-\frac{5}{2}} = \frac{7}{2}, \text{ or } 14x - 4y = 35. \qquad (1)$$

The line through $(7, \frac{11}{2})$ and $(0, \frac{3}{2})$ is

$$\frac{y-\frac{11}{2}}{x-7} = \frac{\frac{3}{2}-\frac{11}{2}}{-7} = \frac{4}{7}, \text{ or } 8x - 14y + 21 = 0. \qquad (2)$$

Regarding (1) and (2) as simultaneous, and solving, we find the intersection to be the point $(\frac{7}{2}, \frac{7}{2})$.

15. Prove that the two straight lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

Let the vertices in order be (a_1, b_1), (a_2, b_2), (a_3, b_3), and (a_4, b_4); then the first mid-point is $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$; the second, $\left(\frac{a_2+a_3}{2}, \frac{b_2+b_3}{2}\right)$; the third, $\left(\frac{a_3+a_4}{2}, \frac{b_3+b_4}{2}\right)$; and the fourth, $\left(\frac{a_1+a_4}{2}, \frac{b_1+b_4}{2}\right)$. Now the mid-point of the line joining the first and third is the same as the mid-point of the line joining the second and fourth, and each is

$$\left(\frac{a_1+a_2+a_3+a_4}{4}, \frac{b_1+b_2+b_3+b_4}{4}\right).$$

16. A line is divided into three equal parts. One end of the line is the point $(3, 8)$; the adjacent point of division is $(4, 13)$. What are the co-ordinates of the other end?

If in formulas [3], (x_1, y_1) is $(3, 8)$, (x, y) is $(4, 13)$, and $m : n$ is $1 : 2$, we have

$$4 = \frac{x_2+6}{3}, \text{ or } x_2 = 6,$$

and
$$13 = \frac{y_2+16}{3}, \text{ or } y_2 = 23.$$

Hence the other end of the line is $(6, 23)$.

17. The line joining the points (x_1, y_1) and (x_2, y_2) is divided into four equal parts. Find the co-ordinates of the points of division.

For the first point of division $m : n$ is $1 : 3$; hence we have

$$x = \frac{x_2 + 3x_1}{4}, \quad y = \frac{y_2 + 3y_1}{4}.$$

For the second point of division $m = n$, and we have

$$x = \frac{x_2 + x_1}{2}, \quad y = \frac{y_2 + y_1}{2}.$$

For the third point of division $m : n$ is $3 : 1$; hence we have

$$x = \frac{3x_2 + x_1}{4}, \quad y = \frac{3y_2 + y_1}{4}.$$

18. Explain and illustrate the relation which exists between an equation and its locus.

The equation expresses the law of motion of the point by which its locus is traced, by stating the relation that exists between its co-ordinates in every one of its different positions.

Or an equation expresses the relation that exists between the co-ordinates of every point of its locus.

19. Construct the two lines which form the locus of the equation $x^2 - 7x = 0$.

The equation may be written in the form

$$x(x - 7) = 0,$$

from which we see that its locus consists of the loci $x = 0$ and $x - 7 = 0$, or $x = 7$. The locus of $x = 0$ is the axis of y, and that of $x = 7$ is a line parallel to the axis of y at the distance 7 to the right.

20. Is the point $(2, -5)$ in the locus of the equation $4x^2 - 9y^2 = 36$?

The co-ordinates of the point $(2, -5)$ do not satisfy the equation $4x^2 - 9y^2 = 36$; hence this point does not lie on the locus of the equation.

21. The ordinate of a certain point in the locus of the equation $x^2 + y^2 + 20x - 70 = 0$ is 1. What is the abscissa of this point?

For $y = 1$ in $x^2 + y^2 + 20x - 70 = 0$ we have

$$x^2 + 20x = 69; \quad \therefore x = 3, \text{ or } -23.$$

22. Find the intercepts of the curve $x^2 + y^2 - 5x - 7y + 6 = 0$.

For $y = 0$ in $x^2 + y^2 - 5x - 7y + 6 = 0$,

$$x = 2 \text{ and } 3.$$

For $x = 0$ in $x^2 + y^2 - 5x - 7y + 6 = 0$,

$$y = 6 \text{ and } 1.$$

Hence the intercepts on the axis of x are 2 and 3, and on the axis of y, 1 and 6.

23. Find the points common to the curves

$$x^2 + y^2 = 100, \text{ and } y^2 - \frac{9x}{2} = 0.$$

Regarding these equations as simultaneous, and solving, we find the four following pairs of values of x and y: $(8, 6)$, $(8, -6)$, $(-\frac{25}{2}, \frac{15}{2}\sqrt{-1})$, $(-\frac{25}{2}, \frac{15}{2}\sqrt{-1})$. Hence the points common to the two curves are $(8, 6)$, and $(8, -6)$; the other pairs of values of x and y do not represent common points, since the value of y in each is imaginary.

24. Find the points common to the curves $x^2 + y^2 = 5a^2$ and $x^2 = 4ay$.

Regarding these equations as simultaneous, and solving, we find the four following pairs of values of x and y: $(2a, a)$, $(-2a, a)$, $(2a\sqrt{-5}, -5a)$, $(-2a\sqrt{-5}, -5a)$, of which the first two only represent common points of the loci.

25. Find the points common to the curves $b^2x^2 + a^2y^2 = a^2b^2$, and $x^2 + y^2 = a^2$.

Regarding these equations as simultaneous, and solving, we find the two following pairs of values of x and y: $(a, 0)$, $(-a, 0)$; hence the curves have the points $(a, 0)$ and $(-a, 0)$ in common.

26. Find the lengths of the sides of a triangle, if its vertices are $(6, 0)$, $(0, -8)$, $(-4, -2)$.

By [1], the distance between $(6, 0)$ and $(0, -8)$ is

$$d = \sqrt{36 + 64} = 10.$$

The distance between $(0, -8)$ and $(-4, -2)$ is

$$d = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}.$$

The distance between $(-4, -2)$ and $(6, 0)$ is

$$d = \sqrt{100 + 4} = \sqrt{104} = 2\sqrt{26}.$$

27. A point moves so that it is always six times as far from one of two fixed perpendicular lines as from the other. Find the equation of its locus.

Let XX' and YY' (Fig. 34) be the two fixed perpendiculars; then, if point $P(x, y)$ is six times as far from OX as from OY, we have $MP = 6\,OM$, or $y = 6x$.

If P' is six times as far from OY as from OX, we have $OM' = 6\,M'P'$, or $x = 6y$.

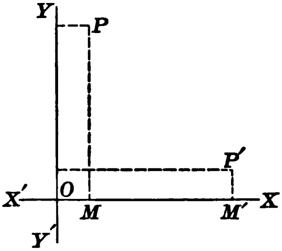

FIG. 34.

28. A point so moves that its distance from the fixed point A is always double its distance from the fixed line AB. Find the equation of its locus.

Take the fixed point A (Fig. 35) as the origin, the fixed line AB as the axis of x, and AY as the axis of y. Let P be any position of the moving point (x, y). Draw PM perpendicular to AB; then from the conditions of the problem we have

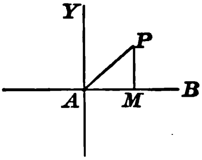

FIG. 35.

$$AP = 2\,MP. \qquad (1)$$

But $AM = x$, $MP = y$, and $AP = \sqrt{AM^2 + MP^2} = \sqrt{x^2 + y^2}$.

Substituting these values in (1), we obtain

$$\sqrt{x^2 + y^2} = 2y, \text{ or } x^2 - 3y^2 = 0.$$

Hence the locus consists of the two straight lines

$$x - y\sqrt{3} = 0 \text{ and } x + y\sqrt{3} = 0.$$

29. A fixed point is at the distance a from a fixed straight line. A point so moves that its distance from the fixed point is always twice its distance from the fixed line. Find the equation of its locus.

Let A (Fig. 36) be the fixed point, and OX the fixed straight line. Take OX as the axis of x, and the perpendicular to it from A as the axis of y. Let P be any position of the moving point (x, y). Draw PM perpendicular to OX, AN parallel to OX, and draw AP. Then from the conditions of the problem we have

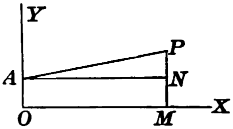

FIG. 36.

$$AP = 2 \times MP, \text{ or } AP^2 = 4 \times MP^2. \qquad (1)$$

But
$$OM = AN = x,\ OA = MN = a,\ MP = y,$$
$$NP = MP - MN = y - a,$$
$$AP^2 = AN^2 + NP^2 = x^2 + (y - a)^2.$$

Substituting these values in (1), we obtain
$$x^2 + (y - a)^2 = 4y^2,$$
or
$$x^2 - 3y^2 - 2ay + a^2 = 0$$
as the required equation.

Ex. 11. Page 40.

1. Find the equation of the straight line passing through the two points (2, 3) and (4, 5).

The equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) is
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad [4]$$

Here (x_1, y_1) is (2, 3), and (x_2, y_2) is (4, 5). Substituting these values in [4], we obtain
$$\frac{y-3}{x-2} = \frac{5-3}{4-2}, \text{ or } x - y + 1 = 0,$$
as the required equation.

2. Find the equation of the straight line passing through the two points (4, 5) and (7, 11).

Here (x_1, y_1) is (4, 5) and (x_2, y_2) is (7, 11). Substituting these values in [4], we obtain
$$\frac{y-5}{x-4} = \frac{11-5}{7-4}, \text{ or } 2x - y - 3 = 0$$
as the required equation.

3. Find the equation of the straight line passing through the two points (−1, 2) and (3, −2).

Here (x_1, y_1) is (−1, 2), and (x_2, y_2) is (3, −2). Substituting these values in [4], we obtain
$$\frac{y-2}{x+1} = \frac{-2, -2}{3+1}, \text{ or } x + y - 1 = 0,$$
as the required equation.

4. Find the equation of the straight line passing through the two points $(-2, -2)$ and $(-3, -3)$.

Here (x_1, y_2) is $(-2, -2)$, and (x_2, y_2) is $(-3, -3)$. Substituting these values in [4], we obtain

$$\frac{y+2}{x+2} = \frac{-3+2}{-3+2}, \text{ or } x - y = 0,$$

as the required equation.

5. Find the equation of the straight line passing through the two points $(4, 0)$ and $(2, 3)$.

Here (x_1, y_1) is $(4, 0)$, and (x_2, y_2) is $(2, 3)$. Substituting these values in [4], we obtain

$$\frac{y}{x-4} = \frac{3}{2-4}, \text{ or } 3x + 2y - 12 = 0,$$

as the required equation.

6. Find the equation of the straight line passing through the two points $(0, 2)$ and $(-3, 0)$.

Here (x_1, y_1) is $(0, 2)$, and (x_2, y_2) is $(-3, 0)$. Substituting these values in [4], we obtain

$$\frac{y-2}{x} = \frac{0-2}{-3}, \text{ or } 2x - 3y + 6 = 0$$

as the required equation.

7. Find the equation of the straight line passing through the two points $(2, 5)$ and $(0, 7)$.

Here (x_1, y_1) is $(2, 5)$, and (x_2, y_2) is $(0, 7)$. Substituting these values in [4], we obtain

$$\frac{y-5}{x-2} = \frac{7-5}{0-2}, \text{ or } x + y - 7 = 0,$$

as the required equation.

8. Find the equation of the straight line passing through the two points $(3, 4)$ and $(0, 0)$.

Here (x_1, y_1) is $(3, 4)$, and (x_2, y_2) is $(0, 0)$. Substituting these values in [4], we obtain

$$\frac{y-4}{x-3} = \frac{0-4}{0-3}, \text{ or } 4x - 3y = 0,$$

as the required equation.

9. Find the equation of the straight line passing through the two points (3, 0) and (0, 0).

Here (x_1, y_1) is (3, 0), and (x_2, y_2) is (0, 0). Substituting these values in [4], we obtain

$$\frac{y}{x-3}=\frac{0}{-3}, \text{ or } y=0,$$

as the required equation.

10. Find the equation of the straight line passing through the two points (3, 4) and (−2, 4).

Here (x_1, y_1) is (3, 4), and (x_2, y_2) is (−2, 4). Substituting these values in [4], we obtain

$$\frac{y-4}{x-3}=\frac{4-4}{-2-3}, \text{ or } y-4=0$$

as the required equation.

11. Find the equation of the straight line passing through the two points (2, 5) and (−2, −5).

Here (x_1, y_1) is (2, 5), and (x_2, y_2) is (−2, −5). Substituting these values in [4], we obtain

$$\frac{y-5}{x-2}=\frac{-5-5}{-2-2}, \text{ or } 5x-2y=0$$

as the required equation.

12. Find the equation of the straight line passing through the two points (m, n) and $(-m, -n)$.

Here (x_1, y_1) is (m, n), and (x_2, y_2) is $(-m, -n)$. Substituting these values in [4], we obtain

$$\frac{y-n}{x-m}=\frac{-n-n}{-m-m}, \text{ or } nx-my=0,$$

as the required equation.

13. Find the equation of a straight line, given (4, 1) and $\gamma=45^\circ$.

The equation of a straight line passing through the point (x_1, y_1) and having the slope m [$=\tan\gamma$] is

$$y-y_1=m(x-x_1). \qquad [5]$$

Here (x_1, y_1) is (4, 1), and $m=\tan\gamma=\tan 45^\circ=1$. Substituting these values in [5], we obtain

$$y-1=x-4, \text{ or } x-y-3=0,$$

as the required equation.

14. Find the equation of a straight line, given (2, 7) and $\gamma = 60^\circ$.

Here (x_1, y_1) is (2, 7), and $m = \tan\gamma = \tan 60^\circ = \sqrt{3}$. Substituting these values in [5], we obtain

$$y - 7 = \sqrt{3}\,(x - 2), \text{ or } \sqrt{3}x - y + 7 - 2\sqrt{3} = 0,$$

as the required equation.

15. Find the equation of a straight line, given (−3, 11) and $\gamma = 45^\circ$.

Here (x_1, y_1) is (−3, 11), and $m = \tan\gamma = \tan 45^\circ = 1$. Substituting these values in [5], we obtain

$$y - 11 = x + 3, \text{ or } x - y + 14 = 0,$$

as the required equation.

16. Find the equation of a straight line, given (13, −4) and $\gamma = 150^\circ$.

Here (x_1, y_1) is (13, −4), and $m = \tan\gamma = \tan 150^\circ = -\frac{1}{3}\sqrt{3}$. Substituting these values in [5], we obtain

$$y + 4 = -\tfrac{1}{3}\sqrt{3}\,(x - 13), \text{ or } \sqrt{3}x + 3y - 13\sqrt{3} + 12 = 0,$$

as the required equation.

17. Find the equation of a straight line, given (3, 0) and $\gamma = 30^\circ$.

Here (x_1, y_1) is (3, 0), and $m = \tan\gamma = \tan 30^\circ = \frac{1}{3}\sqrt{3}$. Substituting these values in [5], we obtain

$$y = \tfrac{1}{3}\sqrt{3}\,(x - 3), \text{ or } \sqrt{3}x - 3y - 3\sqrt{3} = 0,$$

as the required equation.

18. Find the equation of a straight line, given (0, 3) and $\gamma = 135^\circ$.

Here (x_1, y_1) is (0, 3), and $m = \tan\gamma = \tan 135^\circ = -1$. Substituting these values in [5], we obtain

$$y - 3 = -1(x), \text{ or } x + y - 3 = 0,$$

as the required equation.

19. Find the equation of a straight line, given (0, 0) and $\gamma = 120^\circ$.

Here (x_1, y_1) is (0, 0), and $m = \tan\gamma = \tan 120^\circ = -\sqrt{3}$. Substituting these values in [5], we obtain

$$\sqrt{3}x + y = 0,$$

as the required equation.

20. Find the equation of a straight line, given $(2, -3)$ and $\gamma = 0°$.

Here (x_1, y_1) is $(2, -3)$, and $m = \tan\gamma = \tan 0° = 0$. Substituting these values in [5], we obtain

$$y + 3 = 0.$$

as the required equation.

21. Find the equation of the straight line, given $(2, -3)$ and $\gamma = 90°$.

Here (x_1, y_1) is $(2, -3)$, and $m = \tan\gamma = \tan 90° = \infty$. Substituting these values in [5], we obtain

$$y + 3 = \infty (x - 2). \tag{1}$$

Now, in order that (1) may be true, we must have

$$x - 2 = 0,$$

which is the required equation.

Or, writing [5] in the form

$$\frac{x - x_1}{y - y_1} = \frac{1}{m},$$

and substituting the values of x_1, y_1, and m, we obtain

$$\frac{x - 2}{y + 3} = \frac{1}{\infty} = 0, \text{ or } x - 2 = 0,$$

as the required equation.

22. Find the equation of the straight line, given $b = 2$ and $\gamma = 45°$.

The equation of the straight line whose intercept on the axis of y is b, and whose slope is m, is

$$y = mx + b. \tag*{[6]}$$

Here $b = 2$, and $m = \tan\gamma = \tan 45° = 1$. Substituting these values in [6], we obtain

$$y = x + 2,$$

as the required equation.

23. Find the equation of the straight line, given $b = 5$, $\gamma = 45°$.

Here $b = 5$, and $m = \tan\gamma = \tan 45° = 1$. Substituting these values in [6], we obtain

$$y = x + 5,$$

as the required equation.

24. Find the equation of the straight line, given $b = -4$, $\gamma = 45°$.

Here $b = -4$, and $m = \tan\gamma = \tan 45° = 1$. Substituting these values in [6], we obtain

$$y = x - 4,$$

as the required equation.

25. Find the equation of the straight line, given $b = -4$, $\gamma = 30°$.

Here $b = -4$, and $m = \tan\gamma = \tan 30° = \frac{1}{3}\sqrt{3}$. Substituting these values in [6], we obtain

$$y = \tfrac{1}{3}\sqrt{3}\,x - 4,$$

as the required equation.

26. Find the equation of the straight line, given $b = -4$, $\gamma = 0°$.

Here $b = -4$, and $m = \tan\gamma = \tan 0° = 0$. Substituting these values in [6], we obtain

$$y = -4,$$

as the required equation.

27. Find the equation of the straight line, given $b = -4$, $\gamma = 60°$.

Here $b = -4$, and $m = \tan\gamma = \tan 60° = \sqrt{3}$. Substituting these values in [6], we obtain

$$y = \sqrt{3}\,x - 4,$$

as the required equation.

28. Find the equation of the straight line, given $b = -4$, $\gamma = 90°$.

Here $b = -4$, and $m = \tan\gamma = \tan 90° = \infty$. Substituting these values in [6], we obtain

$$y = \infty\, x - 4. \qquad (1)$$

Now, in order that (1) may be true, we must have

$$x = 0,$$

which is therefore the required equation.

Or, writing [6] in the form

$$x = \frac{y}{m} - \frac{b}{m},$$

and substituting the values of m and b, we obtain

$$x = \frac{y}{\infty} + \frac{4}{\infty}, \text{ or } x = 0,$$

as the required equation.

29. Find the equation of the straight line, given $b=-4$, $\gamma=120^\circ$.

Here $b=-4$, and $m=\tan\gamma=\tan 120^\circ=-\sqrt{3}$. Substituting these values in [6], we obtain

$$y=-\sqrt{3}x-4,$$

as the required equation.

30. Find the equation of the straight line, given $b=-4$, $\gamma=135^\circ$.

Here $b=-4$, and $m=\tan\gamma=\tan 135^\circ=-1$. Substituting these values in [6], we obtain

$$y=-x-4,$$

as the required equation.

31. Find the equation of the straight line, given $b=-4$, $\gamma=150^\circ$.

Here $b=-4$, and $m=\tan\gamma=\tan 150^\circ=-\frac{1}{3}\sqrt{3}$. Substituting these values in [6], we obtain

$$y=-\tfrac{1}{3}\sqrt{3}x-4,$$

as the required equation.

32. Find the equation of the straight line, given $b=-4$, $\gamma=180^\circ$.

Here $b=-4$, and $m=\tan\gamma=\tan 180^\circ=0$. Substituting these values in [6], we obtain

$$y=-4,$$

as the required equation.

33. Find the equation of the straight line, given $a=4$, $b=3$.

The equation of the straight line whose intercepts on the axis of x and y are a and b respectively, is

$$\frac{x}{a}+\frac{y}{b}=1. \qquad [7]$$

Here $a=4$, and $b=3$. Substituting these values in [7], we obtain

$$\frac{x}{4}+\frac{y}{3}=1, \text{ or } 3x+4y-12=0,$$

as the required equation.

34. Find the equation of the straight line, given $a=-6$, $b=2$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{-6}+\frac{y}{2}=1, \text{ or } x-3y+6=0,$$

as the required equation.

35. Find the equation of the straight line, given $a = -3$, $b = -3$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{-3} + \frac{y}{-3} = 1, \text{ or } x + y + 3 = 0,$$

as the required equation.

36. Find the equation of the straight line, given $a = 5$, $b = -3$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{5} + \frac{y}{-3} = 1, \text{ or } 3x - 5y - 15 = 0,$$

as the required equation.

37. Find the equation of the straight line, given $a = -10$, $b = 5$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{-10} + \frac{y}{5} = 1, \text{ or } x - 2y + 10 = 0,$$

as the required equation.

38. Find the equation of the straight line, given $a = 1$, $b = -1$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{1} + \frac{y}{-1} = 1, \text{ or } x - y - 1 = 0,$$

as the required equation.

39. Find the equation of the straight line, given $a = n$, $b = -n$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{n} + \frac{y}{-n} = 1, \text{ or } x - y - n = 0,$$

as the required equation.

40. Find the equation of the straight line, given $a = n$, $b = 4n$.

Substituting the values of a and b in [7], we obtain

$$\frac{x}{n} + \frac{y}{4n} = 1, \text{ or } 4x + y - 4n = 0,$$

as the required equation.

41. Find the equation of the straight line, given $p = 5$, $\alpha = 45^\circ$.

The equation of the straight line the perpendicular to which from the origin equals p, and makes the angle α with the axis of x, is

$$x \cos \alpha + y \sin \alpha = p. \qquad [8]$$

Here $p = 5$, $\cos \alpha = \cos 45^\circ = \frac{1}{2}\sqrt{2}$, and $\sin \alpha = \sin 45^\circ = \frac{1}{2}\sqrt{2}$.

Substituting these values in [8], we obtain

$$\tfrac{1}{2}\sqrt{2}x + \tfrac{1}{2}\sqrt{2}y = 5, \text{ or } x + y - 5\sqrt{2} = 0,$$

as the required equation.

42. Find the equation of the straight line, given $p = 5$, $\alpha = 120^\circ$.

Here $p = 5$, $\sin \alpha = \sin 120^\circ = \frac{1}{2}\sqrt{3}$, and $\cos \alpha = \cos 120^\circ = -\frac{1}{2}$.

Substituting these values in [8], we obtain

$$-\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{3}y = 5, \text{ or } x - \sqrt{3}y + 10 = 0,$$

as the equation required.

43. Find the equation of the straight line, given $p = 5$, $\alpha = 240^\circ$.

Here $p = 5$, $\sin \alpha = \sin 240^\circ = -\frac{1}{2}\sqrt{3}$, and $\cos \alpha = \cos 240^\circ = -\frac{1}{2}$.

Substituting these values in [8], we obtain

$$-\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{3}y = 5, \text{ or } x + \sqrt{3}y + 10 = 0,$$

as the required equation.

44. Find the equation of the straight line, given $p = 5$, $\alpha = 300^\circ$.

Here $p = 5$, $\sin \alpha = \sin 300^\circ = -\frac{1}{2}\sqrt{3}$, $\cos \alpha = \cos 300^\circ = \frac{1}{2}$.

Substituting these values in [8], we obtain

$$\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{3}y = 5, \text{ or } x - \sqrt{3}y - 10 = 0,$$

as the required equation.

45. Write the equations of the sides of a triangle if its vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$.

The equation of a straight line passing through any two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad [4]$$

Let (x_1, y_1) be $(2, 1)$, and (x_2, y_2) be $(3, -2)$. Substituting these values in [4], we obtain

$$\frac{y - 1}{x - 2} = \frac{-2 - 1}{3 - 2}, \text{ or } 3x + y - 7 = 0,$$

as the equation of the side between the vertices $(2, 1)$ and $(3, -2)$.

Letting (x_1, y_1) be $(2, 1)$, and (x_2, y_2) be $(-4, -1)$, in [4], we obtain

$$\frac{y-1}{x-2}=\frac{-1-1}{-4-2}, \text{ or } x-3y+1=0,$$

as the equation of the side between the vertices (2, 1) and (− 4, − 1).

In [4], letting (x_1, y_1) be (3, −2), and (x_2, y_2) be (−4, −1), we obtain

$$\frac{y+2}{x-3}=\frac{-1+2}{-4-3}, \text{ or } x+7y+11=0,$$

as the equation of the side between the vertices (3, −2) and (− 4, − 1).

46. Write the equations of the sides of a triangle if its vertices are the points (2, 3), (4, − 5), (−3, − 6).

In [4], letting (x_1, y_1) be (2, 3), and (x_2, y_2) be (4, −5), we obtain

$$\frac{y-3}{x-2}=\frac{-5-3}{4-2}, \text{ or } 4x+y-11=0,$$

as the equation of the side through (2, 3) and (4, − 5).

In [4], letting (x_1, y_1) be (2, 3), and (x_2, y_2) be (−3, −6), we obtain

$$\frac{y-3}{x-2}=\frac{-6-3}{-3-2}, \text{ or } 9x-5y-3=0,$$

as the equation of the side through (2, 3) and (−3, −6).

In [4], letting (x_1, y_1) be (4, −5), and (x_2, y_2) be (− 3, − 6), we obtain

$$\frac{y+5}{x-4}=\frac{-6+5}{-3-4}, \text{ or } x-7y-39=0,$$

as the equation of the side through (4, − 5) and (−3, −6).

47. Form the equations of the medians of the triangle described in No. 46.

If (x, y) be the mid-point between (x_1, y_1) and (x_2, y_2), then

$$x=\frac{x_1+x_2}{2}, \quad y=\frac{y_1+y_2}{2}. \qquad [2]$$

In [2], let (x_1, y_1) be (2, 3), and (x_2, y_2) be (4, −5) ; then

$$x=\frac{2+4}{2}=3, \quad y=\frac{3-5}{2}=-1;$$

that is, the mid-point opposite the vertex (− 3, − 6) is (3, − 1).

Now in [4], letting (x_1, y_1) be $(-3, -6)$, and (x_2, y_2) be $(3, -1)$, we obtain

$$\frac{y+6}{x+3} = \frac{-1+6}{3+3}, \text{ or } 5x - 6y - 21 = 0,$$

as the median from the vertex $(-3, -6)$.

In [2], let (x_1, y_1) be $(4, -5)$, and (x_2, y_2) be $(-3, -6)$; then

$$x = \frac{4-3}{2} = \frac{1}{2}, \quad y = \frac{-5-6}{2} = -\frac{11}{2};$$

that is, the mid-point opposite the vertex $(2, 3)$ is $(\frac{1}{2}, -\frac{11}{2})$.

In [4], letting (x_1, y_1) be $(2, 3)$, and (x_2, y_2) be $(\frac{1}{2}, -\frac{11}{2})$, we obtain

$$\frac{y-3}{x-2} = \frac{-\frac{11}{2}-3}{\frac{1}{2}-2}, \text{ or } 17x - 3y - 25 = 0,$$

as the median from the vertex $(2, 3)$.

In [2] let (x_1, y_1) be $(-3, -6)$, and (x_2, y_2) be $(2, 3)$; then

$$x = \frac{-3+2}{2} = -\frac{1}{2}, \quad y = \frac{-6+3}{2} = -\frac{3}{2};$$

that is, the mid-point opposite the vertex $(4, -5)$ is $(-\frac{1}{2}, -\frac{3}{2})$.

In [4], letting (x_1, y_1) be $(4, -5)$, and (x_2, y_2) be $(-\frac{1}{2}, -\frac{3}{2})$, we obtain

$$\frac{y+5}{x-4} = \frac{-\frac{3}{2}+5}{-\frac{1}{2}-4}, \text{ or } 7x + 9y + 17 = 0,$$

as the median from the vertex $(4, -5)$.

48. The vertices of a quadrilateral are $(0, 0)$, $(1, 5)$, $(7, 0)$, $(4, -9)$. Form the equations of its sides, and also of its diagonals.

By inspection or construction, we learn that the vertices in order are $(0, 0)$, $(1, 5)$, $(7, 0)$, $(4, -9)$.

In [4], letting (x_1, y_1) be $(0, 0)$, and (x_2, y_2) be $(1, 5)$, we obtain

$$\frac{y}{x} = \frac{5}{1}, \text{ or } 5x - y = 0,$$

as the equation of the first side.

In [4], letting (x_1, y_1) be $(1, 5)$, and (x_2, y_2) be $(7, 0)$, we obtain

$$\frac{y-5}{x-1} = \frac{-5}{7-1}, \text{ or } 5x + 6y - 35 = 0,$$

as the equation of the second side.

In [4], letting (x_1, y_1) be (7, 0), and (x_2, y_2) be (4, −9), we obtain

$$\frac{y}{x-7}=\frac{-9}{4-7}, \text{ or } 3x-y-21=0,$$

as the equation of the third side.

In [4], letting (x_1, y_1) be (4, −9), and (x_2, y_2) be (0, 0), we obtain

$$\frac{y+9}{x-4}=\frac{9}{-4}, \text{ or } 9x+4y=0,$$

as the equation of the fourth side.

In [4], letting (x_1, y_1) be (0, 0), and (x_2, y_2) be (7, 0), we obtain

$$\frac{y}{x}=\frac{0}{7}, \text{ or } y=0,$$

as the equation of one diagonal.

In [4], letting (x_1, y_1) be (1, 5), and (x_2, y_2) be (4, −9), we obtain

$$\frac{y-5}{x-1}=\frac{-9-5}{4-1}, \text{ or } 14x+3y-29=0,$$

as the equation of the other diagonal.

49. Find the equation of a straight line, given $a=7\frac{1}{2}$, $\gamma=30°$.

Reducing $\frac{x}{a}+\frac{y}{b}=1$ to the form $y=mx+b$, we have

$$y=-\frac{b}{a}x+b. \tag{1}$$

Here $a=7\frac{1}{2}$, and $-\frac{b}{a}=m=\tan\gamma=\tan 30°=\frac{1}{3}\sqrt{3}$; hence

$$b=-\frac{1}{3}\sqrt{3}\,a=-\frac{5}{2}\sqrt{3}.$$

Substituting these values of a and b in (1) or [7], we obtain

$$2x-2\sqrt{3}\,y-15=0,$$

as the required equation.

Or, since $a=7\frac{1}{2}$, the line must pass through the point $(7\frac{1}{2}, 0)$. Hence (x_1, y_1) is $(7\frac{1}{2}, 0)$, and $m=\tan\gamma=\tan 30°=\frac{1}{3}\sqrt{3}$. Substituting these values in $y-y_1=m(x-x_1)$, we obtain

$$2x-2\sqrt{3}\,y-15=0.$$

50. Find the equation of a straight line, given $a = -3$, (x_1, y_1) is $(2, 5)$.

Since $a = -3$, the line passes through the point $(-3, 0)$; hence (x_2, y_2) is $(-3, 0)$, and (x_1, y_1) is $(2, 5)$. Substituting these values in [4], we obtain

$$\frac{y-5}{x-2} = \frac{-5}{-3-2}, \text{ or } y = x + 3,$$

as the required equation.

51. Find the equation of a straight line, given $p = 6$, $\gamma = 45°$.

Reducing $x \cos \alpha + y \sin \alpha = p$ to the form $y = mx + b$, we have

$$y = -x \cot \alpha + \frac{p}{\sin \alpha}. \qquad (1)$$

Here $p = 6$, and $-\cot \alpha = m = \tan \gamma = \tan 45° = 1$, or $\cot \alpha = -1$; hence $\alpha = 135°$, or $315°$, and $\sin \alpha = \frac{1}{2}\sqrt{2}$, or $-\frac{1}{2}\sqrt{2}$. Substituting these values in (1), we obtain

$$y = x \pm 6\sqrt{2},$$

as the required equations.

52. Find the equation of a straight line, given $p = 6$, $\gamma = 135°$.

Here $p = 6$, and $-\cot \alpha = m = \tan 135° = -1$, or $\cot \alpha = 1$; hence $\alpha = 45°$, or $225°$, and $\sin \alpha = \frac{1}{2}\sqrt{2}$, or $-\frac{1}{2}\sqrt{2}$. Substituting these values in (1) of No. 51, we obtain

$$y = -x \pm 6\sqrt{2},$$

as the required equations.

53. Reduce $3x - 2y + 11 = 0$ and $y = 7x + 1$ to the symmetrical form, and construct each by its intercepts.

The characteristics of the symmetrical form are that the second member is $+1$, and the other constants are the denominators of the terms in x and y.

(i.) Transposing $+11$ and dividing both members by -11, the first equation becomes

$$\frac{3x}{-11} + \frac{2y}{11} = 1; \quad \therefore \frac{x}{-\frac{11}{3}} + \frac{y}{\frac{11}{2}} = 1,$$

which is in the symmetrical form.

Hence $a = -\frac{11}{3}$, and $b = \frac{11}{2}$.

To construct the equation: from O in the direction OX' lay off $OR = \frac{11}{3}$, and from O in the direction OY lay off $OS = \frac{11}{2}$. The straight line through R and S will be the locus required.

(ii.) Since $y = 7x + 1$,

$$-7x + y = 1, \text{ or } \frac{x}{-\frac{1}{7}} + \frac{y}{1} = 1,$$

which is in the symmetrical form.

Hence $a = -\frac{1}{7}$, and $b = 1$.

To construct the equation, lay off these intercepts on the axes, and draw a straight line through their extremities.

54. Reduce $3x + 5y - 13 = 0$ and $4x - y - 2 = 0$, to the symmetrical form, and construct each by its intercepts.

In the first equation, transposing -13 and dividing by 13, we obtain

$$\frac{3x}{13} + \frac{5y}{13} = 1; \quad \therefore \frac{x}{\frac{13}{3}} + \frac{y}{\frac{13}{5}} = 1,$$

which is in the symmetrical form.

Hence $a = \frac{13}{3}$, and $b = \frac{13}{5}$.

Lay off these intercepts on the axes, and draw a straight line through their extremities.

In the second equation, transposing -2 and dividing by 2, we obtain

$$2x - \frac{y}{2} = 1; \quad \therefore \frac{x}{\frac{1}{2}} + \frac{y}{-2} = 1.$$

Hence $a = \frac{1}{2}$, and $b = -2$.

Lay off these intercepts on the axes, and draw a straight line through their extremities.

55. Reduce $Ax + By = C$ to the symmetrical form; also $y = mx + b$. What are the values of a and b in terms of A, B, C, and m?

Dividing $Ax + By = C$ by C, we have

$$\frac{Ax}{C} + \frac{By}{C} = 1; \quad \therefore \frac{x}{C \div A} + \frac{y}{C \div B} = 1.$$

Hence $a = C \div A$, and $b = C \div B$.

In $y = mx + b$, transposing mx and dividing by b, we have

$$\frac{y}{b} - \frac{mx}{b} = 1; \quad \therefore \frac{x}{-(b \div m)} + \frac{y}{b} = 1.$$

Hence $a = -\frac{b}{m}$, and $b = b$.

56. Reduce $y+13=5x$ and $y+19=7x$ to the form $y=mx+b$, and construct each by its slope, and intercept on the axis of y.

Solving $y+13=5x$ for y, we obtain

$$y=5x+(-13), \qquad (1)$$

which is in the form of $y=mx+b$.

Hence $m=5$, and $b=-13$.

To construct (1), lay off $OB=-13$, draw BA parallel to OX, lay off $BA=+1$, draw AP parallel to OY, lay off $AP=5$; the straight line through B and P will be the locus of (1).

For the intercept of BD is $OB=-13$, and its slope $=\tan XCD=\tan ABP=AP\div BA=5$.

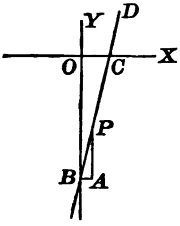

Fig. 37.

Solving $y+19=7x$ for y, we obtain,

$$y=7x-19. \qquad (2)$$

Hence $m=7$ and $b=-19$.

To construct (2), lay off $OB=-19$, $BA=1$, $AP=7$, and draw BP, which will be the required locus.

57. Reduce $3x+y+2=0$ and $2y=3x+6$ to the form $y=mx+b$, and construct each by its slope, and intercept on the axis of y.

Solving $3x+y+2=0$ for y, we obtain

$$y=-3x-2. \qquad (1)$$

Hence $m=-3$, and $b=-2$.

To construct (1), lay off $OB=-2$, $BA=1$, $AP=-2$; then BP will be the locus of (1).

For the intercept of $BP=OB=-2$, and its slope $=\tan XRH=\tan ABR=-\tan PBA=-(AP\div BA)=-3$.

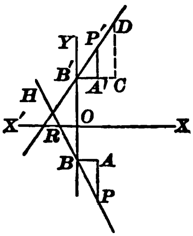

Fig. 38.

Solving $2y=3x+6$ for y, we obtain

$$y=\tfrac{3}{2}x+3. \qquad (2)$$

Hence $m=\tfrac{3}{2}$, and $b=3$.

To construct (2), lay off $OB'=3$, draw $B'A'$ parallel to OX, lay off $B'A'=1$, draw $A'P'$ parallel to OY, and lay off $A'P'=\tfrac{3}{2}$; then $B'P'$ will be the locus of (2).

For its intercept $=3$, and its slope $=A'P'\div B'A'=\tfrac{3}{2}$.

To determine the slope, we might take $B'C=2$, and $CD=3$.

58. Reduce $Ax + By = C$ to the form $y = mx + b$; also $\frac{x}{a} + \frac{y}{b} = 1$. What are the values of m and b in terms of A, B, C, and a?

Solving $Ax + By = C$ for y, we obtain

$$y = -(A \div B)\,x + (C \div B).$$

Hence $m = -(A \div B)$, and $b = C \div B$.

Solving $\frac{x}{a} + \frac{y}{b} = 1$ for y, we obtain

$$y = -\frac{b}{a}x + b.$$

Hence $m = -(b \div a)$ and $b = b$.

59. Find the vertices of the triangle whose sides are the lines

$$2x + 9y + 17 = 0,\ y = 7x - 38,\ 2y - x = 2.$$

Regarding $2x + 9y + 17 = 0$ and $y = 7x - 38$ as simultaneous, and solving for x and y, we obtain

$$x = 5, \text{ and } y = -3.$$

Hence, the vertex of the angle between these sides is $(5, -3)$.

Regarding $y = 7x - 38$ and $2y - x = 2$ as simultaneous, and solving for x and y, we find the vertex of the second angle to be $(6, 4)$.

Regarding $2y - x = 2$ and $2x + 9y + 17 = 0$ as simultaneous, and solving for x and y, we find the vertex of the third angle to be $(-4, -1)$.

60. Find the equation of the straight line passing through the origin and the intersection of the lines $3x - 2y + 4 = 0$ and $3x + 4y = 5$. Also find the distance between these two points.

Solving $3x - 2y + 4 = 0$ and $3x + 4y = 5$, as simultaneous equations, we find that the common point of their loci is $(-\frac{1}{3}, \frac{3}{2})$.

Let (x_1, y_1) be $(-\frac{1}{3}, \frac{3}{2})$, and (x_2, y_2) be $(0, 0)$. Substituting these values in [4], we obtain

$$\frac{y - \frac{3}{2}}{x + \frac{1}{3}} = \frac{-\frac{3}{2}}{\frac{1}{3}}, \text{ or } 9x + 2y = 0,$$

as the required equation.

Substituting these values of x_1, y_1, x_2, y_2 in [1], we obtain $\frac{1}{6}\sqrt{85}$ as the required distance.

61. What is the equation of the line passing through (x_1, y_1), and equally inclined to the two axes?

The line with its intercepts on the axes form a right triangle, whose acute angles are each 45°; hence, if the line crosses the first or the third quadrant, its slope will be tan 135°, or -1. If it crosses the second or the fourth quadrant, its slope must be tan 45°, or 1.

The equation of a line through (x_1, y_1) having the slope m is

$$y - y_1 = m(x - x_1).$$

Here $m = \mp 1$; hence the required equations are

$$y - y_1 = \mp (x - x_1), \text{ or } y \pm x = y_1 \pm x_1.$$

62. Find the equations of the diagonals of the parallelogram formed by the lines $x = a$, $x = b$, $y = c$, $y = d$.

The intersections of the line $x = a$ with the lines $y = c$ and $y = d$, are (a, c) and (a, d) respectively; and the intersections of $x = b$ with $y = c$ and $y = d$, are (b, c) and (b, d) respectively. Hence the vertices in order are (a, c), (a, d), (b, d), and (b, c).

Let (x_1, y_1) be (a, c), and (x_2, y_2) be (b, d). Substituting these values in [4], we obtain

$$\frac{y-c}{x-a} = \frac{d-c}{b-a}, \text{ or } (d-c)x - (b-a)y = ad - bc,$$

as the equation of the diagonal through (a, c) and (b, d).

In [4], letting (x_1, y_1) be (a, d), and (x_2, y_2) be (b, c), we obtain

$$\frac{y-d}{x-a} = \frac{c-d}{b-a}, \text{ or } (d-c)x + (b-a)y = bd - ac,$$

as the equation of the other diagonal.

63. Show that the lines $y = 2x + 3$, $y = 3x + 4$, $y = 4x + 5$ all pass through one point.

Solving $y = 2x + 3$ and $y = 3x + 4$ as simultaneous equations, we find that the common point of their loci is $(-1, 1)$. But $x = -1$, and $y = 1$, satisfy $y = 4x + 5$; hence $(-1, 1)$, the intersection of the first two lines, lies upon the third.

64. The vertices of a triangle are $(0, 0)$, $(x_1, 0)$, (x_2, y_2). Find the equations of its medians, and prove that they meet in one point.

The mid-point of the side between $(0, 0)$ and $(x_1, 0)$ is evidently $(\frac{1}{2}x_1, 0)$.

Let (x_1, y_1) be $(\frac{1}{2}x_1, 0)$, and (x_2, y_2) be (x_2, y_2). Substituting these values in [4], we obtain

$$\frac{y}{x - \frac{1}{2}x_1} = \frac{y_2}{x_2 - \frac{1}{2}x_1}, \text{ or } 2y_2x + (x_1 - 2x_2)y = x_1y_2, \quad (1)$$

as the equation of the median through (x_2, y_2).

If in [2] we put $y_1 = 0$, we find for the mid-point between $(x_1, 0)$ and (x_2, y_2)

$$x = \frac{x_1 + x_2}{2}, \quad y = \tfrac{1}{2} y_2.$$

Now let (x_1, y_1) be $\left(\frac{x_1 + x_2}{2}, \frac{1}{2} y_2\right)$, and (x_2, y_2) be $(0, 0)$. Substituting these values in [4], we obtain

$$\frac{y - \frac{1}{2} y_2}{x - \frac{1}{2}(x_1 + x_2)} = \frac{-\frac{1}{2} y_2}{-\frac{1}{2}(x_1 + x_2)}, \text{ or } y_2 x - (x_1 + x_2) y = 0, \qquad (2)$$

as the equation of the median through $(0, 0)$.

The mid-point between $(0, 0)$ and (x_2, y_2) is evidently $(\frac{1}{2} x_2, \frac{1}{2} y_2)$.

Letting (x_1, y_1) be $(x_1, 0)$, and (x_2, y_2) be $(\frac{1}{2} x_2, \frac{1}{2} y_2)$, in [4], we obtain

$$\frac{y}{x - x_1} = \frac{\frac{1}{2} y_2}{\frac{1}{2} x_2 - x_1}, \text{ or } y_2 x + (2 x_1 - x_2) y = x_1 y_2, \qquad (3)$$

as the equation of the median through the vertex $(x_1, 0)$.

Solving (1) and (2) as simultaneous equations, we obtain as the intersection of their loci the point $\left(\frac{x_1 + x_2}{3}, \frac{y_2}{3}\right)$. But the co-ordinates of this point satisfy (3); hence the locus of (3) passes through the intersection of (1) and (2).

65. What must be the value of m if the line $y = mx$ passes through the point $(1, 4)$?

If the line $y = mx$ passes through the point $(1, 4)$, its co-ordinates must satisfy the equation $y = mx$.

Substituting these co-ordinates, we obtain

$$4 = m, \text{ or } m = 4.$$

66. The line $y = mx + 3$ passes through the intersection of the lines $y = x + 1$ and $y = 2x + 2$. Determine the value of m.

Solving $y = x + 1$ and $y = 2x + 2$ as simultaneous, we find $x = -1$ and $y = 0$. Hence the intersection of their loci is the point $(-1, 0)$.

Substituting the co-ordinates of this point in $y = mx + 3$, we obtain $m = 3$.

67. Find the value of b, if the line $y = 6x + b$ passes through the point $(2, 3)$.

Putting $x = 2$, and $y = 3$ in the equation $y = 6x + b$, we obtain $b = -9$.

68. What condition must be satisfied if the points (x_1, y_1), (x_2, y_2) (x_3, y_3) lie in one straight line?

If (x_3, y_3) be upon the line through (x_1, y_1) and (x_2, y_2), then $x = x_3$ and $y = y_3$ must satisfy [4]. Hence we have

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ or } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

69. Discuss equation [5] for the following cases: (i.) (x_1, y_1) is $(0, 0)$, (ii.) $m = 0$, (iii.) $m = \infty$.

(i.) If (x_1, y_1) is $(0, 0)$, [5] becomes $y = mx$, the locus of which passes through the origin, since it has no constant term.

(ii.) If $m = 0$, [5] becomes $y - y_1 = 0$, or $y = y_1$, the locus of which is a straight line parallel to the axis of x.

(iii.) If $m = \infty$, [5] becomes $y - y_1 = \infty\,(x - x_1)$, which is satisfied only when $x - x_1 = 0$, or $x = x_1$, the locus of which is a line parallel to the axis of y.

70. Discuss equation [6] for the following cases: (i.) $b = 0$, (ii.) $m = 0$, (iii.) $m = \infty$, (iv.) $m = 0$ and $b = 0$.

(i.) If $b = 0$, [6] becomes $y = mx$, and its locus passes through the origin.

(ii.) If $m = 0$, [6] becomes $y = b$, and its locus is parallel to the axis of x.

(iii.) If in [6] $b = mc$ and $m = \infty$, we have $y = \infty\,(x + c)$, which is satisfied only when $x + c = 0$; hence the locus is parallel to the axis of y.

(iv.) If $m = 0$, and $b = 0$, [6] becomes $y = 0$, and its locus is the axis of x.

71. Discuss equation [7] for the following cases: (i.) $a = b$, (ii.) $a = 0$, (iii.) $a = \infty$, (iv.) $b = \infty$.

(i.) If $a = b$, [7] becomes $y = -x + a$, whose locus crosses the first or the third quadrant, according as a is positive or negative.

(ii.) If $a = 0$, $b = 0$, and [7] becomes $\frac{x}{0} + \frac{y}{0} = 1$, or $y = -\frac{0}{0}x$; hence the locus is a line through the origin.

(iii.) If $a = \infty$, [7] becomes $\frac{x}{\infty} + \frac{y}{b} = 1$, which is satisfied only when $y = b$; hence the locus is a line parallel to the axis of x.

(iv.) If $b = \infty$, [7] becomes $\frac{x}{a} + \frac{y}{\infty} = 1$, which is satisfied only when $x = a$; hence the locus is a line parallel to the axis of y.

Ex. 12. Page 44.

1. Reduce $3x - 2y + 11 = 0$ to the normal form, and thus determine p, or the distance of the locus from the origin.

To put this equation in the form of [9], we transpose $+11$ and divide both members by -1, and obtain

$$-3x + 2y = 11. \tag{1}$$

Dividing both members of (1) by $\sqrt{(-3)^2 + 2^2}$, or $\sqrt{13}$, we have

$$\frac{-3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y = \frac{11}{\sqrt{13}},$$

which is in the normal form.

Hence $p = \frac{11}{\sqrt{13}}$, $\cos\alpha = \frac{-3}{\sqrt{13}}$, and $\sin\alpha = \frac{2}{\sqrt{13}}$.

2. Reduce $3x + 5y - 13 = 0$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing -13, we obtain

$$3x + 5y = 13, \tag{1}$$

which is in the form of [9].

Dividing (1) by $\sqrt{3^2 + 5^2}$, or $\sqrt{34}$, we obtain

$$\frac{3}{\sqrt{34}}x + \frac{5}{\sqrt{34}}y = \frac{13}{\sqrt{34}},$$

which is in the normal form.

Hence $\cos\alpha = \frac{3}{\sqrt{34}}$, $\sin\alpha = \frac{5}{\sqrt{34}}$, and $p = \frac{13}{\sqrt{34}}$.

3. Reduce $4x - y - 2 = 0$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing -2, and dividing by $\sqrt{4^2 + (-1)^2}$, or $\sqrt{17}$, we have

$$\frac{4}{\sqrt{17}}x - \frac{1}{\sqrt{17}}y = \frac{2}{\sqrt{17}},$$

which is in the normal form. Hence $p = \frac{2}{17}\sqrt{17}$.

4. Reduce $2x + 3y = 7$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Dividing both members by $\sqrt{2^2 + 3^2}$, or $\sqrt{13}$, we obtain

$$\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{7}{\sqrt{13}},$$

which is in the normal form. Hence $p = \frac{7}{13}\sqrt{13}$.

5. Reduce $y + 13 = 5x$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing $5x$ and $+13$, and dividing by -1, we have

$$5x - y = 13. \qquad (1)$$

Dividing by $\sqrt{5^2 + (-1)^2}$, or $\sqrt{26}$, we have

$$\frac{5}{\sqrt{26}}x - \frac{1}{\sqrt{26}}y = \frac{13}{\sqrt{26}},$$

which is in the normal form. Hence $p = \frac{1}{2}\sqrt{26}$.

6. Reduce $y + 19 = 7x$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing $7x$ and $+19$, and dividing by -1, we have

$$7x - y = 19.$$

Dividing by $\sqrt{50}$, or $5\sqrt{2}$, we have

$$\frac{7}{5\sqrt{2}}x - \frac{1}{5\sqrt{2}}y = \frac{19}{5\sqrt{2}},$$

which is in the normal form. Hence $p = 1.9 \times \sqrt{2}$.

7. Reduce $ex + cy + n = 0$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing $+n$, and dividing by -1, we obtain

$$-ex - cy = n.$$

Dividing by $\sqrt{e^2 + c^2}$, we have

$$-\frac{ex}{\sqrt{e^2 + c^2}} - \frac{cy}{\sqrt{e^2 + c^2}} = \frac{n}{\sqrt{e^2 + c^2}},$$

which is in the normal form when n is positive.

Hence $\cos \alpha = \frac{-e}{\sqrt{e^2 + c^2}}$, $\sin \alpha = \frac{-c}{\sqrt{e^2 + c^2}}$, $p = \frac{n}{\sqrt{e^2 + c^2}}$.

8. Reduce $ny + cx - r = 0$ to the normal form, and thus determine p, or the distance of the locus from the origin.

Transposing $-r$, and dividing by $\sqrt{n^2 + c^2}$, we have

$$\frac{cx}{\sqrt{n^2 + c^2}} + \frac{ny}{\sqrt{n^2 + c^2}} = \frac{r}{\sqrt{n^2 + c^2}},$$

which is in the normal form. Hence $p = r \div (n^2 + c^2)^{\frac{1}{2}}$.

9. By the signs of the constants in $y = \frac{1}{3}x - 9$ determine which of the four quadrants its locus crosses.

Here $m = \frac{1}{3}$, and $b = -9$. Since b is negative, the line cuts the axis of y below the origin; and since m is positive, the line inclines to the right, and hence crosses the fourth quadrant.

10. Reduce $3x + 2 = 2y$ to the form of [6], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [6], we have

$$y = \tfrac{3}{2}x + 1.$$

Here $m = \frac{3}{2}$, and $b = 1$. Since b is positive, the locus cuts the axis of y above the origin; and since m is positive, the line inclines to the right. Hence the line crosses the second quadrant.

11. Reduce $4y = 5x - 1$ to the form of [7], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [7], we obtain

$$\frac{x}{\frac{1}{5}} + \frac{y}{-\frac{1}{4}} = 1.$$

Hence $a = \frac{1}{5}$, and $b = -\frac{1}{4}$. Therefore the locus crosses the fourth quadrant.

12. Reduce $4y = 3x + 24$ to the form of [7], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [7], we obtain

$$\frac{x}{-8} + \frac{y}{6} = 1.$$

Hence $a = -8$, and $b = 6$. Therefore the locus crosses the second quadrant.

13. Reduce $5x+3y+15=0$ to the form of [8], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing the equation to the form of [8], we obtain

$$\frac{-5}{\sqrt{34}}x-\frac{3}{\sqrt{34}}y=\frac{15}{\sqrt{34}}.$$

Since $\sin\alpha$ and $\cos\alpha$ are each negative, α is of the third quadrant; hence the locus crosses the third quadrant.

14. Reduce $5x+4y-20=0$ to the form of [8], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing the equation to the form of [8], we obtain

$$\frac{5}{\sqrt{41}}x+\frac{4}{\sqrt{41}}y=\frac{20}{\sqrt{41}}.$$

Since $\sin\alpha$ and $\cos\alpha$ are each positive, α is of the first quadrant; hence the locus crosses the first quadrant.

15. By the signs of the constants in $y=6x+12$, determine which of the four quadrants its locus crosses.

Here $m=6$, and $b=12$; hence the locus cuts the axis of y above the origin and inclines to the right. Therefore the locus crosses the second quadrant.

16. Reduce $y+2=x-4$ to the form of [7], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [7], we have

$$\frac{x}{6}+\frac{y}{-6}=1.$$

Hence $a=6$, and $b=-6$, and therefore the locus crosses the fourth quadrant.

17. Reduce $x+\sqrt{3}y+10=0$ to the form of [8], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [8], we have

$$-\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{3}\,y=5.$$

Since $\cos\alpha$ and $\sin\alpha$ are each negative, α is of the third quadrant, and hence the locus crosses the third quadrant.

18. Reduce $x - \sqrt{3}\,y - 10 = 0$ to the form of [8], and determine by the signs of the constants which of the four quadrants its locus crosses.

Reducing this equation to the form of [8], we have

$$\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{3}\,y = 5.$$

Since $\cos a$ is +, and $\sin a$ is −, a is of the fourth quadrant, and hence the locus crosses the fourth quadrant.

19. Discuss equation [9], or $Ax + By = C$, for the following cases:

(i.) $A = 0$. (iv.) $A = \infty$. (vii.) $A = B,\ C = 0$.
(ii.) $B = 0$. (v.) $A = C = 0$. (viii.) $A = -B,\ C = 0$.
(iii.) $C = 0$. (vi.) $A = B$.

(i.) If $A = 0$, [9] becomes $y = C \div B$; hence the locus is parallel to the axis of x.

(ii.) If $B = 0$, [9] becomes $x = C \div A$; hence the locus is parallel to the axis of y.

(iii.) If $C = 0$, [9] becomes $y = -(A \div B)\,x$; hence the locus is a line through the origin, having the slope $-(A \div B)$.

(iv.) If $A = \infty$, [9] becomes $\infty x + By = C$, which is satisfied only when $x = 0$; hence the locus is the axis of y.

(v.) If $A = C = 0$, [9] becomes $By = 0$; hence the locus is the axis of x.

(vi.) If $A = B$, [9] becomes $y = -x + (C \div A)$; hence the locus makes an angle 135° with the axis of x.

(vii.) If $A = B$, $C = 0$, [9] becomes $y = -x$; hence the locus bisects the second and fourth quadrants.

(viii.) If $A = -B$, $C = 0$, [9] becomes $y = x$; hence the locus bisects the first and third quadrants.

20. Reduce equation [7] to the form of equation [6], and find the value of m in terms of a and b.

Reducing [7] to the form of $y = mx + b$, we have

$$y = -\frac{b}{a}x + b;$$

hence $m = -(b \div a)$.

21. What value must C have in order that the line $4x - 5y = C$ may pass through the origin? Through (2, 0)?

If the locus of $4x - 5y = C$ passes through the origin, the equation cannot have any constant term; hence $C = 0$.

If the locus passes through the point (2, 0), its co-ordinates must satisfy $4x - 5y = C$. Putting $x = 2$, and $y = 0$, we obtain $C = 8$.

22. Determine the values of A, B, and C, so that the line $Ax + By = C$ may pass through (3, 0) and (0, −12).

If the locus passes through the points (3, 0) and (0, −12), their co-ordinates must satisfy $Ax + By = C$. Hence we must have $3A = C$, and $-12B = C$.

Whence $A = \frac{1}{3}C$, and $B = -\frac{1}{12}C$.

Substituting these values of A and B in $Ax + By = C$, and dividing by C, we obtain

$$4x - y = 12,$$

as the required equation.

Hence $A = 4$, $B = -1$, and $C = 12$ is our set of values of A, B, C.

23. From [9] deduce [4] by the method used in No. 22.

If the locus of

$$Ax + By = C \tag{1}$$

passes through (x_1, y_1) and (x_2, y_2), we must have

$$Ax_1 + By_1 = C, \tag{2}$$

and

$$Ax_2 + By_2 = C. \tag{3}$$

Subtracting (2) from (1), we obtain

$$A(x - x_1) + B(y - y_1) = 0, \text{ or } \frac{y - y_1}{x - x_1} = -\frac{A}{B}. \tag{4}$$

Subtracting (2) from (3), we obtain

$$A(x_2 - x_1) + B(y_2 - y_1) = 0, \text{ or } \frac{y_2 - y_1}{x_2 - x_1} = -\frac{A}{B}. \tag{5}$$

From (4) and (5) we obtain

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

24. If equations [4] and [9] represent the same line, what are the values of A, B, C, in terms of x_1, y_1, x_2, and y_2?

Reducing [4] to the form of [9] by clearing of fractions, etc., we obtain

$$(y_2 - y_1)\,x - (x_2 - x_1)\,y = x_1y_2 - x_2y_1. \tag{1}$$

Comparing (1) with

$$Ax + By = C, \qquad [9]$$

we have $A = y_2 - y_1,\ B = -(x_2 - x_1),\ C = x_1y_2 - x_2y_1.$

25. In equation [4] find the values of m and b in terms of x_1, y_1, x_2, y_2.

Reducing (1) in No. 24 to the form of $y = mx + b$, we have

$$y = \frac{y_2 - y_1}{x_2 - x_1}\,x + \frac{x_2y_1 - x_1y_2}{x_2 - x_1}.$$

Hence $m = \dfrac{y_2 - y_1}{x_2 - x_1},\quad b = \dfrac{x_2y_1 - x_1y_2}{x_2 - x_1}.$

Ex. 13. Page 46.

1. Find the equation of the straight line passing through $(3, -7)$, and parallel to the line $y = 3x - 5$.

The equation of the straight line through (x_1, y_1) parallel to the line $y = mx + b$ is

$$y - y_1 = m\,(x - x_1). \qquad [5]$$

Here (x_1, y_1) is $(3, -7)$, and $m = 3$. Substituting these values in [5], we obtain

$$y + 7 = 3\,(x - 3),\ \text{or}\ y = 3x - 16,$$

as the required equation.

2. Find the equation of the straight line passing through $(5, 3)$, and parallel to the line $\frac{1}{3}y - \frac{1}{4}x = 1$.

Here (x_1, y_1) is $(5, 3)$, and $y = mx + b$ is $y = \frac{3}{4}x + 3$, or $m = \frac{3}{4}$. Substituting these values in [5] of No. 1, we obtain

$$y - 3 = \tfrac{3}{4}(x - 5),\ \text{or}\ y = \tfrac{3}{4}x - \tfrac{3}{4},$$

as the required equation.

3. Find the equation of the straight line passing through $(0, 0)$, and parallel to the line $y - 4x = 10$.

Here (x_1, y_1) is $(0, 0)$, and $y = mx + b$ is $y = 4x + 10$, or $m = 4$. Substituting these values in [5] of No. 1, we obtain

$$y = 4x,$$

as the required equation.

4. Find the equation of the straight line passing through (5, 8), and parallel to the axis of x.

Here (x_1, y_1) is (5, 8), and $y = mx + b$ is $y = 0$, or $m = 0$. Substituting these values in [5], we obtain

$$y - 8 = 0, \text{ or } y = 8,$$

as the required equation.

5. Find the equation of the straight line passing through (5, 8), and parallel to the axis of y.

Here (x_1, y_1) is (5, 8), and $y = mx + b$ is $y = \infty x$, or $m = \infty$. Substituting these values in [5], we obtain

$$y - 8 = \infty (x - 5),$$

which is satisfied only when $x - 5 = 0$; hence $x - 5 = 0$, or $x = 5$, is the required equation.

Or, the equation of any line parallel to the axis of y is $x = a$, and if this line passes through (5, 8), $a = 5$; hence $x = 5$ is the required equation.

6. Find the equation of the straight line passing through (3, −13), and perpendicular to the line $y = 4x - 7$.

The equation of the straight line through (x_1, y_1) perpendicular to the line $y = mx + b$ is

$$y - y_1 = -\frac{1}{m}(x - x_1). \tag{B}$$

Here (x_1, y_1) is (3, −13), and $y = mx + b$ is $y = 4x - 7$, or $m = 4$. Substituting these values in (B), we obtain

$$y + 13 = -\tfrac{1}{4}(x - 3), \text{ or } y = -\tfrac{1}{4}x - 12\tfrac{1}{4},$$

as the required equation.

7. Find the equation of the straight line passing through (2, 9), and perpendicular to the line $7y + 23x - 5 = 0$.

Here (x_1, y_1) is (2, 9), and $y = mx + b$ is $y = -\frac{23}{7}x + \frac{5}{7}$, or $m = -\frac{23}{7}$. Substituting these values in (B) of No. 6, we obtain

$$y - 9 = \tfrac{7}{23}(x - 2), \text{ or } y = \tfrac{7}{23}x + 8\tfrac{9}{23},$$

as the required equation.

8. Find the equation of the straight line passing through (0, 0), and perpendicular to the line $x + 2y = 1$.

Here (x_1, y_1) is $(0, 0)$, and $y = mx + b$ is $y = -\frac{1}{2}x + \frac{1}{2}$, or $m = -\frac{1}{2}$. Substituting these values in (B) of No. 6, we obtain

$$y = 2x,$$

as the required equation.

9. Find the equation of the straight line perpendicular to the line $5x - 7y + 1 = 0$, and erected at the point whose abscissa $= 1$.

For $x = 1$ in $5x - 7y + 1 = 0$, $y = \frac{6}{7}$.

Hence (x_1, y_1) is $(1, \frac{6}{7})$, and $y = mx + b$ is $y = \frac{5}{7}x + \frac{1}{7}$, or $m = \frac{5}{7}$. Substituting these values in (B) of No. 6, we obtain

$$y - \frac{6}{7} = -\frac{7}{5}(x - 1), \text{ or } y = -\frac{7}{5}x + 2\tfrac{9}{35},$$

as the required equation.

Ex. 14. Page 47.

1. Find the angle formed by the lines

$$x + 2y + 1 = 0 \text{ and } x - 3y - 4 = 0.$$

If ϕ denote the angle formed by the lines $y = mx + b$ and $y = m'x + b'$, then

$$\tan\phi = \frac{m - m'}{1 + mm'}. \qquad [10]$$

Let $y = mx + b$ be $y = -\frac{1}{2}x - \frac{1}{2}$, and $y = m'x + b'$ be $y = \frac{1}{3}x - \frac{4}{3}$; then $m = -\frac{1}{2}$, and $m' = \frac{1}{3}$. Substituting these values in [10], we obtain

$$\tan\phi = \frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}} = -1, \text{ or } \phi = 135°.$$

If we let $y = mx + b$ be $y = \frac{1}{3}x - \frac{4}{3}$, and $y = m'x + b'$ be $y = -\frac{1}{2}x - \frac{1}{2}$, we obtain

$$\tan\phi = 1, \text{ or } \phi = 45°.$$

The two answers are supplements of each other, and either gives the other also.

2. Find the tangent of the angle formed by the lines

$$3x - 4y = 7 \text{ and } 2x - y = 3.$$

Let $y = mx + b$ be $y = \frac{3}{4}x - \frac{7}{4}$, and $y = m'x + b'$ be $y = 2x - 3$; then $m = \frac{3}{4}$, and $m' = 2$. Whence

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{\frac{3}{4} - 2}{1 + \frac{6}{4}} = -0.5.$$

3. Find the tangent of the angle formed by the lines

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0.$$

Let $y = mx + b$ be $y = -\frac{2}{3}x - \frac{4}{3}$, and $y = m'x + b'$ be $y = -\frac{3}{4}x - \frac{5}{4}$; then $m = -\frac{2}{3}$ and $m' = -\frac{3}{4}$. Whence

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{-\frac{2}{3} + \frac{3}{4}}{1 + \frac{6}{12}} = \frac{1}{18}.$$

4. Find the tangent of the angle formed by the lines

$$y - nx = 1 \text{ and } 2(y - 1) = nx.$$

Let $y = mx + b$ be $y = nx + 1$, and $y = m'x + b'$ be $y = \frac{n}{2}x + 1$; then $m = n$, and $m' = \frac{1}{2}n$. Whence

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{n - \frac{1}{2}n}{1 + \frac{1}{2}n^2} = \frac{n}{2 + n^2}.$$

5. Find the angle formed by the lines

$$x + y = 1 \text{ and } y = x + 4..$$

Let $y = mx + b$ be $y = -x + 1$, and $y = m'x + b'$ be $y = x + 4$; then $m = -1$, $m' = +1$. Whence the lines are perpendicular, since their slopes are negative reciprocals. (§ 45, Cor. 2.)

6. Find the angle formed by the lines

$$y + 3 = 2x \text{ and } y + 3x = 2.$$

Let $y = mx + b$ be $y = 2x - 3$, and $y = m'x + b'$ be $y = -3x + 2$; then $m = 2$, $m' = -3$. Whence

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{5}{-5} = -1. \quad \therefore \phi = 135^\circ.$$

7. Find the angle formed by the lines

$$2x + 3y + 7 = 0 \text{ and } 3x - 2y + 4 = 0.$$

Let $y = mx + b$ be $y = -\frac{2}{3}x - \frac{7}{3}$, and $y = m'x + b'$ be $y = \frac{3}{2}x + 2$; then $m = -\frac{2}{3}$, $m' = \frac{3}{2}$. Whence the lines are perpendicular, since their slopes are negative reciprocals.

8. Find the angle formed by the lines

$$6x = 2y + 3 \text{ and } y - 3x = 10.$$

Let $y = mx + b$ be $y = 3x - \frac{3}{2}$, and $y = m'x + b'$ be $y = 3x + 10$; then $m = 3 = m'$. Whence the lines are parallel, since their slopes are equal. (§ 45, Cor. 1.)

9. Find the angle formed by the lines

$$x + 3 = 0 \text{ and } y - \sqrt{3}x + 4 = 0.$$

Let $y = mx + b$ be $x + 3 = 0$, or $y = \infty(x + 3)$, and $y = m'x + b'$ be $y = \sqrt{3}x - 4$; then $m = \infty$ and $m' = \sqrt{3}$. Hence

$$\tan\phi = \frac{\infty - \sqrt{3}}{1 + \sqrt{3}\infty} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}. \quad \therefore \phi = 30^\circ.$$

10. Discuss equation [11] for the cases where $\phi = 0^\circ$ and $\phi = 90^\circ$.

If in

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}(x - x_1), \qquad [11]$$

$\phi = 0$, or $\tan\phi = 0$, [11] becomes

$$y - y_1 = m(x - x_1),$$

the locus of which, by § 46, is parallel to $y = mx + b$, as it should be.

If in [11], $\phi = 90^\circ$, or $\tan\phi = \infty$, we have

$$y - y_1 = \frac{m \pm \infty}{1 \mp m\infty}(x - x_1) = -\frac{1}{m}(x - x_1),$$

the locus of which, by § 46, is perpendicular to $y = mx + b$, as it should be.

11. Find the equation of a straight line passing through the point (3, 5), and making the angle 45° with the line $2x - 3y + 5 = 0$.

The equation of the straight line through (x_1, y_1) making the angle ϕ with the line $y = mx + b$ is

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}(x - x_1). \qquad [11]$$

Here (x_1, y_1) is (3, 5), $\tan\phi = \tan 45^\circ = 1$, and $y = mx + b$ is $y = \frac{2}{3}x + \frac{5}{3}$, or $m = \frac{2}{3}$. Substituting these values in [11], we obtain as the required equations

$$y - 5 = \frac{\frac{2}{3} \pm 1}{1 \mp \frac{2}{3}}(x - 3),$$

or

$$y = 5x - 10, \text{ and } y = -\tfrac{1}{5}x + 5\tfrac{3}{5},$$

the loci of which are evidently perpendicular to each other.

12. Find the equation of a straight line passing through the point $(-2, 1)$, and making the angle 45° with the line $2y = 6 - 3x$.

Here (x_1, y_1) is $(-2, 1)$, $\tan\phi = \tan 45^\circ = 1$, and $y = mx + b$ is $y = -\frac{3}{2}x + 3$, or $m = -\frac{3}{2}$. Substituting these values in [11], we obtain as the equations of the lines,

$$y - 1 = \frac{-\frac{3}{2} \pm 1}{1 \mp (-\frac{3}{2})}(x + 2),$$

$$y = -\tfrac{1}{5}x + \tfrac{3}{5}, \text{ and } y = 5x + 11.$$

13. Find the equation of the line passing through that point of the line $y = 2x - 1$ for which $x = 2$, and making the angle 30° with the same line.

For $x = 2$ in $y = 2x - 1$, $y = 3$. Hence (x_1, y_1) is $(2, 3)$, $\tan\phi = \tan 30^\circ = \frac{1}{3}\sqrt{3}$, and $y = mx + b$ is $y = 2x - 1$, or $m = 2$. Substituting these values in [11], we obtain as the equations of the lines,

$$y - 3 = \frac{2 \pm \frac{1}{3}\sqrt{3}}{1 \mp \frac{2}{3}\sqrt{3}}(x - 2),$$

or
$$y - 3 = -(8 \pm 5\sqrt{3})(x - 2).$$

14. Find the equation of the line passing through $(1, 3)$, and making the angle 30° with the line $x - 2y + 1 = 0$.

Here (x_1, y_1) is $(1, 3)$, $\tan\phi = \tan 30^\circ = \frac{1}{3}\sqrt{3}$, and $y = mx + b$ is $y = \frac{1}{2}x + \frac{1}{2}$, or $m = \frac{1}{2}$. Substituting these values in [11], we obtain as the equations of the lines,

$$y - 3 = \frac{\frac{1}{2} \pm \frac{1}{3}\sqrt{3}}{1 \mp \frac{1}{6}\sqrt{3}}(x - 1),$$

or
$$y - 3 = \frac{8 \pm 5\sqrt{3}}{11}(x - 1).$$

15. Prove that the lines represented by the equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

are parallel if $AB' = A'B$; perpendicular, if $AA' = -BB'$.

Reducing these equations to the form $y = mx + b$, we have

$$y = -\frac{A}{B}x - \frac{C}{B}, \quad y = -\frac{A'}{B'}x - \frac{C'}{B'}.$$

If their loci are parallel, their slopes are equal. (§ 45, Cor. 1.)

That is, $$-\frac{A}{B} = -\frac{A'}{B'}, \text{ or } AB' = A'B.$$

If their loci are perpendicular, by § 45, Cor. 2, we must have

$$\left(-\frac{A}{B}\right)\left(-\frac{A'}{B'}\right) + 1 = 0. \quad \therefore AA' = -BB'.$$

16. Given the equation $3x + 4y + 6 = 0$; show that the general equations representing (i.) all parallels and (ii.) all perpendiculars to the given line are

(i.) $3x + 4y + K = 0.$

(ii.) $4x - 3y + K = 0.$

The slope of $$3x + 4y + 6 = 0 \qquad (1)$$
is $-\frac{3}{4}$.

The slope of $$3x + 4y + K = 0 \qquad (2)$$
is $-\frac{3}{4}$, and its intercept is $-(K \div 4)$.

The slope of $$4x - 3y + K = 0 \qquad (3)$$
is $\frac{4}{3}$, and its intercept is $K \div 3$.

Now (1) and (2) are parallel, since their slopes are equal. Moreover, since the intercept $-(K \div 4)$ may have any value, (2) represents all parallels to (1).

Since the slopes of (3) and (1) are negative reciprocals, (3) is perpendicular to (1); and since $K \div 3$ may have any value, (3) represents all perpendiculars to (1).

17. Deduce the following equations for lines passing through (x_1, y_1) and (i.) parallel, (ii.) perpendicular, to the line $y = mx + b$.

$$\text{(i.) } y - mx = y_1 - mx_1.$$
$$\text{(ii.) } my + x = my_1 + x_1.$$

(i.) Let the equation of the line parallel to $y = mx + b$ be
$$y = mx + b'. \qquad (1)$$

Since (1) is to pass through (x_1, y_1), we must have
$$y_1 = mx_1 + b'. \qquad (2)$$

Equating the values of b' as obtained from (1) and (2), we obtain
$$y - mx = y_1 - mx_1.$$

(ii.) Let the equation of the line perpendicular to $y = mx + b$ be
$$y = -\frac{1}{m}x + b'. \qquad (3)$$

Since (3) passes through (x_1, y_1), we must have
$$y_1 = -\frac{1}{m}x_1 + b'. \qquad (4)$$

Equating the values of mb' as obtained from (1) and (2), we obtain
$$my + x = my_1 + x_1.$$

18. Write the equations of 3 lines parallel, and 3 lines perpendicular, to the line $2x + 3y + 1 = 0$.

The slope of each of the lines
$$4x + 6y + 9 = 0, \quad 6x + 9y - 10 = 0, \quad 10x + 15y - 42 = 0,$$

is $-\frac{2}{3}$, which is the slope of $2x+3y+1=0$. Hence all these lines are parallel.

The slope of each of the lines

$$6x-4y-14=0,\quad 9x-6y-14=0,\quad 15x-10y+50=0,$$

is $\frac{3}{2}$, which is the negative reciprocal of the slope of $2x+3y+1=0$. Hence these three lines are perpendicular to the given line.

19. Among the following lines select parallel lines; perpendicular lines; lines neither parallel nor perpendicular:

(i.) $2x+3y-1=0$.
(ii.) $3x-2y=20$.
(iii.) $4x+6y=0$.
(iv.) $12x=8y+7$.
(v.) $x-y=2$.
(vi.) $5(x+y)-11=0$.
(vii.) $x=8$.
(viii.) $y+10=0$.

Line	(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)
Slope	$-\frac{2}{3}$,	$+\frac{3}{2}$,	$-\frac{2}{3}$,	$+\frac{3}{2}$,	$+1$,	-1,	∞,	0.

Hence lines (i.) and (iii.) are parallel, also (ii.) and (iv.); lines (i.) and (iii.) are both perpendicular to (ii.) and (iv); (v.) is perpendicular to (vi.), and (vii.) to (viii.). (§ 45, Cors. 1 and 2.)

Neither (i.), (ii.), (iii.), nor (iv.) is perpendicular or parallel to (v.), (vi.), (vii.), or (viii.), nor is (v.) or (vi.) parallel or perpendicular to (vii.) or (viii.).

20. Prove that the angle ϕ, between the lines

$$Ax+By+C=0 \quad\text{and}\quad A'x+B'y+C'=0,$$

is determined by the equation

$$\tan\phi=\frac{A'B-AB'}{AA'+BB'}.$$

Let $\quad y=mx+b$ be $y=-\frac{A}{B}x-\frac{C}{B}$, or $m=-\frac{A}{B}$,

and $\quad y=m'x+b'$ be $y=-\frac{A'}{B'}x-\frac{C'}{B'}$, or $m'=-\frac{A'}{B'}$.

$$\therefore \tan\phi=\frac{m-m'}{1+mm'}=\frac{-\frac{A}{B}+\frac{A'}{B'}}{1+\frac{AA'}{BB'}}=\frac{A'B-AB'}{AA'+BB'}.$$

21. From the preceding equation deduce the conditions of parallel lines and perpendicular lines given in No. 15.

In No. 20 if the lines are parallel, $\tan\phi = 0$; hence $A'B - AB' = 0$, or $A'B = AB'$, and conversely.

If these lines are perpendicular, $\tan\phi = \infty$; hence $AA' + BB' = 0$, or $AA' = -BB'$, and conversely.

22. Find the equation of a straight line parallel to $2x + 3y + 6 = 0$, passing through (5, 7).

The equation of a line through (x_1, y_1) parallel to $y = mx + b$ is

$$y - y_1 = m(x - x_1). \qquad [5]$$

Here (x_1, y_1) is (5, 7), and $y = mx + b$ is $y = -\frac{2}{3}x - 2$, or $m = -\frac{2}{3}$.

Substituting these values in [5], we obtain as the required equation

$$y - 7 = -\tfrac{2}{3}(x - 5), \text{ or } y = -\tfrac{2}{3}x + 10\tfrac{1}{3}.$$

23. Find the equation of a straight line parallel to $2x + y - 1 = 0$, and passing through the intersection of $3x + 2y - 59 = 0$ and $5x - 7y + 6 = 0$.

Solving $3x + 2y - 59 = 0$ and $5x - 7y + 6 = 0$ as simultaneous, we obtain $x = \frac{401}{31}$ and $y = \frac{313}{31}$; hence their common point (x_1, y_1) is $(\frac{401}{31}, \frac{313}{31})$.

Also $y = mx + b$ is $y = -2x + 1$, or $m = -2$.

Substituting these values in [5], we obtain

$$y - \tfrac{313}{31} = -2\left(x - \tfrac{401}{31}\right), \text{ or } y = -2x + 35\tfrac{30}{31},$$

as the required equation.

24. Find the equation of the straight line parallel to the line joining (−2, 7) and (−4, −5), and passing through (5, 3).

Let (x_1, y_1) be (−2, 7), and (x_2, y_2) be (−4, −5); then, by [4], the equation of the line through these two points is

$$\frac{y - 7}{x + 2} = \frac{-5 - 7}{-4 + 2}, \text{ or } y = 6x + 19.$$

Hence $m = 6$. Also (x_1, y_1) is (5, 3). Substituting these values in [5], we obtain

$$y - 3 = 6(x - 5), \text{ or } y = 6x - 27,$$

as the required equation.

25. Find the equation of a straight line parallel to $y = mx + b$, and at a distance d from the origin.

Since the line is parallel to $y = mx + b$, its slope is m. Hence let its equation be

$$y = mx + b'. \tag{1}$$

Reducing (1) to the normal form, we have

$$\frac{-mx}{\sqrt{1+m^2}} + \frac{y}{\sqrt{1+m^2}} = \frac{b'}{\sqrt{1+m^2}}. \tag{2}$$

Hence $\frac{b'}{\sqrt{1+m^2}} = p = d$, or $b' = d\sqrt{1+m^2}$.

Substituting this value of b' in (1), we obtain

$$y = mx + d\sqrt{1+m^2},$$

as the equation of the parallel which cuts the axis of y above the origin. It is evident that the equation of the other parallel is

$$y = mx - d\sqrt{1+m^2}.$$

26. Find the equation of a straight line perpendicular to $Ax + By + C = 0$, and cutting an intercept b on the axis of y.

The slope of $Ax + By + C = 0$ is $-(A \div B)$; hence the slope of the perpendicular must be $B \div A$. (§ 45, Cor. 2.) But the intercept of the perpendicular is b; hence the equation of this perpendicular is

$$y = \frac{B}{A}x + b.$$

27. Find the equation of a straight line perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$, and passing through (a, b).

The equation of a line through (x_1, y_1) perpendicular to $y = mx + b$ is

$$y - y_1 = -\frac{1}{m}(x - x_1). \tag{B}$$

Here (x_1, y_1) is the point (a, b), and $y = mx + b$ is $y = -\frac{b}{a}x + b$, or $m = -\frac{b}{a}$.

Substituting these values in (B), we obtain

$$y - b = \frac{a}{b}(x - a), \text{ or } y = \frac{a}{b}x + \frac{b^2 - a^2}{b}.$$

28. Find the equation of a straight line making the angle 45° with $\frac{x}{a}+\frac{y}{b}=1$, and passing through $(a, 0)$.

The equation of the straight line through (x_1, y_1), making the angle ϕ with the line $y = mx + b$, is

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}\,(x - x_1). \qquad [11]$$

Here (x_1, y_1) is $(a, 0)$, $\tan\phi = \tan 45° = 1$; and $y = mx + b$ is $y = -\frac{b}{a}x + b$, or $m = -\frac{b}{a}$.

Substituting these values in [11], we obtain

$$y = \frac{-\frac{b}{a} \pm 1}{1 \pm \frac{b}{a}}\,(x - a),$$

or
$$(a \pm b)\,y + (b \mp a)\,(x - a) = 0,$$

as the equation required.

29. Show that the triangle whose vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$ is a right triangle.

Let (x_1, y_1) be $(2, 1)$, and (x_2, y_2) be $(3, -2)$. Substituting these values in [4], we obtain as the equation of the first side,

$$\frac{y-1}{x-2} = \frac{-2-1}{3-2}, \text{ or } y = -3x + 7. \qquad (1)$$

Let (x_1, y_1) be $(3, -2)$, and (x_2, y_2) be $(-4, -1)$. Substituting in [4], we obtain as the second side

$$\frac{y+2}{x-3} = \frac{-1+2}{-4-3}, \text{ or } y = -\tfrac{1}{7}x - \tfrac{11}{7}. \qquad (2)$$

Let (x_1, y_1) be $(-4, -1)$, and (x_2, y_2) be $(2, 1)$. Substituting in [4], we obtain as the third side,

$$\frac{y+1}{x+4} = \frac{1+1}{2+4}, \text{ or } y = \tfrac{1}{3}x + \tfrac{1}{3}. \qquad (3)$$

Now, by § 45, Cor. 2, lines (1) and (3) are perpendicular. Moreover, (2) is oblique to both (1) and (3). Hence the three lines form a right triangle.

30. The vertices of a triangle are $(-1, -1)$, $(-3, 5)$, $(7, 11)$. Find the equations of its altitudes. Prove that the altitudes meet in one point.

Let (x_1, y_1) be $(-1, -1)$, and (x_2, y_2) be $(-3, 5)$. Substituting in [4], we obtain as the equation of one side,

$$\frac{y+1}{x+1} = \frac{5+1}{-3+1}, \text{ or } y = -3x - 4.$$

The equation of the perpendicular from (x_1, y_1) to $y = mx + b$ is

$$y - y_1 = -\frac{1}{m}(x - x_1). \tag{A}$$

Let (x_1, y_1) be $(7, 11)$, and $y = mx + b$ be $y = -3x - 4$, or $m = -3$. Substituting these values in (A), we obtain

$$y = \tfrac{1}{3}x + \tfrac{26}{3}, \tag{1}$$

as the equation of the altitude through the vertex $(7, 11)$.

Let (x_1, y_1) be $(-3, 5)$, and (x_2, y_2) be $(7, 11)$. Substituting in [4], we obtain as the second side,

$$\frac{y-5}{x+3} = \frac{11-5}{7+3}, \text{ or } y = \tfrac{3}{5}x + \tfrac{34}{5}.$$

Let (x_1, y_1) be $(-1, -1)$, and $y = mx + b$ be $y = \frac{3}{5}x + \frac{34}{5}$, or $m = \frac{3}{5}$. Substituting these values in (A), we obtain

$$y = -\tfrac{5}{3}x - \tfrac{8}{3}, \tag{2}$$

as the altitude through $(-1, -1)$.

Let (x_1, y_1) be $(7, 11)$, and (x_2, y_2) be $(-1, -1)$. Substituting in [4], we obtain as the third side,

$$\frac{y-11}{x-7} = \frac{-1-11}{-1-7}, \text{ or } y = \tfrac{3}{2}x + \tfrac{1}{2}.$$

Let (x_1, y_1) be $(-3, 5)$, and $y = mx + b$ be $y = \frac{3}{2}x + \frac{1}{2}$, or $m = \frac{3}{2}$. Substituting in (A), we obtain as the altitude through $(-3, 5)$,

$$y = -\tfrac{2}{3}x + 3. \tag{3}$$

Solving (1) and (2) as simultaneous, we find the intersection of their loci to be the point $(-\frac{17}{3}, \frac{61}{9})$, whose co-ordinates satisfy (3). Hence the three altitudes meet in a point.

31. Find the equation of the perpendicular erected at the middle point of the line joining $(5, 2)$ to the intersection of $x + 2y - 11 = 0$ and $9x - 2y - 59 = 0$.

Regarding $x+2y-11=0$ and $9x-2y-59=0$ as simultaneous, and solving, we find the intersection of their loci to be (7, 2).

Now from [2] the mid-point between (5, 2) and (7, 2) is (6, 2). And the line through (5, 2) and (7, 2) is evidently $y=2$.

Hence the required perpendicular passes through the point (6, 2), and is parallel to the axis of y. Whence its equation is $x=6$.

32. Find the equations of the perpendiculars erected at the middle points of the sides of the triangle whose vertices are (5, −7), (1, 11), (−4, 13). Prove that these perpendiculars meet in one point.

From [2], the mid-point between (5, −7) and (1, 11) is (3, 2).

From [4], the line through (5, −7), (x_1, y_1), and (1, 11), (x_2, y_2) is

$$\frac{y+7}{x-5}=\frac{11+7}{1-5}, \text{ or } y=-\tfrac{9}{2}x+15\tfrac{1}{2}. \tag{1}$$

By § 46, the line through (3, 2) perpendicular to line (1) is

$$y-2=\tfrac{2}{9}(x-3), \text{ or } y=\tfrac{2}{9}x+\tfrac{4}{3}. \tag{2}$$

From [2], the mid-point between (1, 11) and (−4, 13) is $(-\frac{3}{2}, 12)$.

From [4], the line through (1, 11) and (−4, 13) is

$$\frac{y-11}{x-1}=\frac{13-11}{-4-1}, \text{ or } y=-\tfrac{2}{5}x+11\tfrac{2}{5}. \tag{3}$$

By § 46, the line through $(-\frac{3}{2}, 12)$ perpendicular to line (3) is

$$y-12=\tfrac{5}{2}(x+\tfrac{3}{2}), \text{ or } y=\tfrac{5}{2}x+15\tfrac{3}{4}. \tag{4}$$

From [2], the mid-point between (−4, 13) and (5, −7) is $(\frac{1}{2}, 3)$.

From [4], the line through (−4, 13) and (5, −7) is

$$\frac{y-13}{x+4}=\frac{-7-13}{5+4}, \text{ or } y=-\tfrac{20}{9}x+4\tfrac{1}{9}. \tag{5}$$

By § 46, the line through $(\frac{1}{2}, 3)$ perpendicular to line 5 is

$$y-3=\tfrac{9}{20}(x-\tfrac{1}{2}), \text{ or } y=\tfrac{9}{20}x+\tfrac{111}{40}. \tag{6}$$

Hence (2), (4), and (6) are the equations of the perpendiculars to the sides at their middle points.

Solving (2) and (4) as simultaneous equations, we find the common point of their loci to be $(-\frac{519}{82}, -\frac{8}{41})$, whose co-ordinates satisfy (6).

Hence the three perpendiculars meet in a point.

33. The equations of the sides of a triangle are

$$x+y+1=0, \quad 3x+5y+11=0, \quad x+2y+4=0.$$

Find (1) the equations of the perpendiculars erected at the middle points of the sides; (ii.) the co-ordinates of their common point of intersection; (iii.) the distance of this point from the vertices of the triangle.

(i.) Solving $x + y + 1 = 0$ and $3x + 5y + 11 = 0$ as simultaneous equations, we find the vertex of the angle between their loci to be $(3, -4)$.

Solving $3x + 5y + 11 = 0$ and $x + 2y + 4 = 0$ as simultaneous, we find the second vertex to be $(-2, -1)$.

Solving $x + 2y + 4 = 0$ and $x + y + 1 = 0$ as simultaneous, we find the third vertex to be $(2, -3)$.

Hence the extremities of the side $x + y + 1 = 0$ are $(3, -4)$ and $(2, -3)$, and by [2] its mid-point is $(\frac{5}{2}, -\frac{7}{2})$.

By § 46, the line through $(\frac{5}{2}, -\frac{7}{2})$ perpendicular to $x + y + 1 = 0$, or $y = -x - 1$, is

$$y + \tfrac{7}{2} = x - \tfrac{5}{2}, \text{ or } y = x - 6. \tag{1}$$

The extremities of the side $3x + 5y + 11 = 0$ are $(3, -4)$ and $(-2, -1)$, and by [2] its mid-point is $(\frac{1}{2}, -\frac{5}{2})$.

By § 46, the line through $(\frac{1}{2}, -\frac{5}{2})$ perpendicular to $3x + 5y + 11 = 0$, or $y = -\frac{3}{5}x - \frac{11}{5}$, is

$$y + \tfrac{5}{2} = \tfrac{5}{3}(x - \tfrac{1}{2}), \text{ or } y = \tfrac{5}{3}x - 3\tfrac{1}{3}. \tag{2}$$

The ends of the side $x + 2y + 4 = 0$ are $(-2, -1)$ and $(2, -3)$, and by [2] its mid-point is $(0, -2)$.

By § 46, the line through $(0, -2)$ perpendicular to $y = -\frac{1}{2}x - 2$ is

$$y + 2 = 2x, \text{ or } y = 2x - 2. \tag{3}$$

(ii.) The intersection of (1) and (2) is $(-4, -10)$, and this point lies in (3).

(iii.) The distance of $(-4, -10)$ from $(3, -4)$ is $\sqrt{85}$.
The distance of $(-4, -10)$ from $(-2, -1)$ is $\sqrt{85}$.
The distance of $(-4, -10)$ from $(2, -3)$ is $\sqrt{85}$.

34. Show that the straight line passing through (a, b) and (c, d) is perpendicular to the straight line passing through $(b, -a)$ and $(d, -c)$.

By [4], the straight line through (a, b), (x_1, y_1), and (c, d), (x_2, y_2), is

$$\frac{y - b}{x - a} = \frac{d - b}{c - a}, \tag{1}$$

and the straight line through $(b, -a)$, (x_1, y_1), and $(d, -c)$, (x_2, y_2), is

$$\frac{y + a}{x - b} = \frac{-c + a}{d - b} = -\frac{c - a}{d - b}. \tag{2}$$

The loci of (1) and (2) are perpendicular, since their second members, which are the slopes of their loci, are negative reciprocals. (§ 45, Cor. 2.)

35. What is the equation of a straight line passing through (x_1, y_1), and making an angle ϕ with the line $Ax + By + C = 0$?

The equation of the straight line through (x_1, y_1), making the angle ϕ with the line $y = mx + b$, is

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}(x - x_1). \qquad [11]$$

Here $y = mx + b$ is $y = -\frac{A}{B}x - \frac{C}{B}$, or $m = -\frac{A}{B}$.

Substituting this value of m in [11], we obtain

$$y - y_1 = \frac{-A \pm B\tan\phi}{B \pm A\tan\phi}(x - x_1),$$

as the required equation.

Ex. 15. Page 52.

1. Find the distance from (1, 13) to the line $3x = y - 5$.

By § 46, the equation of the perpendicular from (1, 13) to the line $y = 3x + 5$ is

$$y - 13 = -\tfrac{1}{3}(x - 1), \text{ or } y = -\tfrac{1}{3}x + 13\tfrac{1}{3}. \qquad (1)$$

The intersection of $y = 3x + 5$ and (1) is $(2\frac{1}{2}, 12\frac{1}{2})$.

By [1], the distance from (1, 13) to $(2\frac{1}{2}, 12\frac{1}{2})$ is

$$\sqrt{(1 - 2\tfrac{1}{2})^2 + (13 - 12\tfrac{1}{2})^2}, \text{ or } \tfrac{1}{2}\sqrt{10}.$$

2. Find the distance from (8, 4) to the line $y = 2x - 16$.

By § 46, the equation of the perpendicular from (8, 4) to $y = 2x - 16$ is

$$y - 4 = -\tfrac{1}{2}(x - 8), \text{ or } y = -\tfrac{1}{2}x + 8. \qquad (1)$$

The intersection of $y = 2x - 16$ and (1) is $(\frac{48}{5}, \frac{16}{5})$.

By [1], the distance from (8, 4) to $(\frac{48}{5}, \frac{16}{5})$ is

$$\sqrt{(8 - \tfrac{48}{5})^2 + (4 - \tfrac{16}{5})^2}, \text{ or } \tfrac{4}{5}\sqrt{5}.$$

3. Find the distance from the origin to the line $3x + 4y = 20$.

Reducing $3x + 4y = 20$ to the normal form, we have

$$\tfrac{3}{5}x + \tfrac{4}{5}y = 4.$$

Hence p, or the distance of the line from the origin, is 4.

4. Find the distance from (2, 3) to the line $2x + y - 4 = 0$.

Here $Ax + By = C$ is $2x + y = 4$, and (x_1, y_1) is (2, 3).

Hence $$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}} = \frac{4 + 3 - 4}{\sqrt{5}} = +\tfrac{3}{5}\sqrt{5}.$$

The line lies between the point and the origin.

5. Find the distance from (3, 3) to the line $y = 4x - 9$.

Here $Ax + By = C$ is $4x - y = 9$, and (x_1, y_1) is (3, 3).

Hence $$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}} = \frac{12 - 3 - 9}{\sqrt{17}} = 0.$$

That is, the point is on the line.

6. Prove that the distance from the point (x_1, y_1) to the line $y = mx + b$ is

$$d = \pm\frac{y_1 - mx_1 - b}{\sqrt{1 + m^2}},$$

the upper or lower sign being used according as b is positive or negative. Express this result in the form of a rule for practice.

Here $Ax + By - C = 0$ is $\pm(y - mx - b) = 0$, the upper or lower sign being used according as b is positive or negative.

Hence $$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}} = \pm\frac{y_1 - mx_1 - b}{\sqrt{m^2 + 1}},$$

in which the upper or lower sign is used according as b is positive or negative.

The distance from the point (x_1, y_1) to the line $y = mx + b$ is obtained by substituting x_1 for x, and y_1 for y, in the expression $\pm(y - mx - b)$, and dividing the result by $\sqrt{m^2 + 1}$, the upper or lower sign being used according as b is positive or negative. The point and origin will be on opposite sides of the line, or on the same side, according as the result is positive or negative.

7. Find the distances from the line $3x + 4y + 15 = 0$ to the following points: (3, 0), (3, −1), (3, −2), (3, −3), (3, −4), (3, −5), (3, −6), (3, −7), (0, 0), (−1, 0), (−2, 0), (−3, 0), (−4, 0), (−5, 0), (−6, 0).

Here $Ax + By = C$ is

$$-3x - 4y = 15. \qquad (1)$$

Hence the distance from (x_1, y_1) to (1) is

$$d = \frac{-3x_1 - 4y_1 - 15}{5}. \tag{2}$$

If (x_1, y_1) is $(3, 0)$, $d = -4\frac{4}{5}$.
If (x_1, y_1) is $(3, -1)$, $d = -4$.
If (x_1, y_1) is $(3, -2)$, $d = -3\frac{1}{5}$.
If (x_1, y_1) is $(3, -3)$, $d = -2\frac{2}{5}$.
If (x_1, y_1) is $(3, -4)$, $d = -1\frac{3}{5}$.
If (x_1, y_1) is $(3, -5)$, $d = -\frac{4}{5}$.
If (x_1, y_1) is $(3, -6)$, $d = 0$.
If (x_1, y_1) is $(3, -7)$, $d = +\frac{4}{5}$.

If (x_1, y_1) is $(0, 0)$, $d = -3$.
If (x_1, y_1) is $(-1, 0)$, $d = -2\frac{2}{5}$.
If (x_1, y_1) is $(-2, 0)$, $d = -1\frac{4}{5}$.
If (x_1, y_1) is $(-3, 0)$, $d = -\frac{6}{5}$.
If (x_1, y_1) is $(-4, 0)$, $d = -\frac{3}{5}$.
If (x_1, y_1) is $(-5, 0)$, $d = 0$.
If (x_1, y_1) is $(-6, 0)$, $d = +\frac{3}{5}$.

8. Find the distances from (1, 3) to the following lines:

$3x + 4y + 15 = 0.$
$3x + 4y + 10 = 0.$
$3x + 4y + 5 = 0.$
$3x + 4y = 0.$
$3x + 4y - 5 = 0.$
$3x + 4y - 10 = 0.$
$3x + 4y - 15 = 0.$
$3x + 4y - 20 = 0.$

(i.) $Ax + By = C$ is $-3x - 4y = 15$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}} = \frac{-3 - 12 - 15}{5} = -6.$$

(ii.) $Ax + By = C$ is $-3x - 4y = 10$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}} = \frac{-3 - 12 - 10}{5} = -5.$$

(iii.) $Ax + By = C$ is $-3x - 4y = 5$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{-3 - 12 - 5}{5} = -4.$$

(iv.) $Ax + By = C$ is $3x + 4y = 0$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{3 + 12}{5} = 3.$$

(v.) $Ax + By = C$ is $3x + 4y = 5$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{3 + 12 - 5}{5} = +2.$$

(vi.) $Ax + By = C$ is $3x + 4y = 10$, and (x_1, y_1) is $(1, 3)$.

Hence $$d = \frac{3 + 12 - 10}{5} = +1.$$

(vii.) $Ax + By = C$ is $3x + 4y = 15$, and (x_1, y_1) is (1, 3).

Hence $$d = \frac{3 + 12 - 15}{5} = 0.$$

(viii.) $Ax + By = C$ is $3x + 4y = 20$, and (x_1, y_1) is (1, 3)

Hence $$d = \frac{3 + 12 - 20}{5} = -1.$$

9. Find the distance from the point (2, −5) to the line $y - 3x = 7$.

Here $Ax + By = C$ is $-3x + y = 7$, and (x_1, y_1) is (2, −5).

Hence $$d = \frac{-6 - 5 - 7}{\sqrt{10}} = -\tfrac{9}{5}\sqrt{10}.$$

10. Find the distance from the point (4, 5) to the line $4y + 5x = 20$.

Here $Ax + By = C$ is $5x + 4y = 20$, and (x_1, y_1) is (4, 5).

Hence $$d = \frac{20 + 20 - 20}{\sqrt{41}} = \tfrac{20}{41}\sqrt{41}.$$

11. Find the distance from the point (2, 3) to the line $x + y = 1$.

Here $Ax + By = C$ is $x + y = 1$, and (x_1, y_1) is (2, 3).

Hence $$d = \frac{2 + 3 - 1}{\sqrt{2}} = 2\sqrt{2}.$$

12. Find the distance from the point (0, 1) to the line $3x - 3y = 1$.

Here $Ax + By = C$ is $3x - 3y = 1$, and (x_1, y_1) is (0, 1).

Hence $$d = \frac{-3 - 1}{\sqrt{18}} = -\tfrac{2}{3}\sqrt{2}.$$

13. Find the distance from the point (−1, 3) to the line

$$3x + 4y + 2 = 0.$$

Here $Ax + By = C$ is $-3x - 4y = 2$, and (x_1, y_1) is (−1, 3).

Hence $$d = \frac{3 - 12 - 2}{5} = -2\tfrac{1}{5}.$$

14. Find the distance from the origin to the line $3x + 2y - 6 = 0$.

Reducing the equation to the normal form, we have

$$\frac{3x}{\sqrt{13}} + \frac{2y}{\sqrt{13}} = \frac{6}{\sqrt{13}} = \text{distance from the origin.}$$

15. From the point $(2, -7)$ to the line joining $(-4, 1)$ and $(3, 2)$.

From [4], the equation of the line through $(-4, 1)$, (x_1, y_1) and $(3, 2)$, (x_2, y_2) is

$$\frac{y-1}{x+4}=\frac{2-1}{3+4}, \text{ or } x-7y+11=0.$$

Hence, $Ax+By=C$ is $-x+7y=11$, and (x_1, y_1) is $(2, -7)$.

$$\therefore\ d=\frac{-2-49-11}{5\sqrt{2}}=-\tfrac{31}{5}\sqrt{2}.$$

16. From the line $y=7x$ to the intersection of the lines $y=3x-4$ and $y=5x+2$.

By solving $y=3x-4$ and $y=5x+2$ as simultaneous equations, we find the intersection of their loci to be the point $(-3, -13)$.

Now $Ax+By=C$ is $\pm(7x-y)=0$, and (x_1, y_1) is $(-3, -13)$.

Hence $$d=\frac{\pm(-21+13)}{5\sqrt{2}}=\mp\tfrac{4}{5}\sqrt{2}.$$

17. From the origin to the line $a(x-a)+b(y-b)=0$.

Reducing the equation to the normal form, we have

$$\frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}=p.$$

18. From the points (a, b) and $(-a, -b)$ to the line $\frac{x}{a}+\frac{y}{b}=1$.

Here $Ax+By-C$ is $\pm(bx+ay-ab)$, according as a and b have like or unlike signs, and (x_1, y) is (a, b).

Hence, $$d=\frac{\pm(ab+ab-ab)}{\sqrt{a^2+b^2}}=\frac{\pm ab}{\sqrt{a^2+b^2}},$$

the upper or lower sign being used according as a and b have like or unlike signs.

If (x_1, y_1) be $(-a, -b)$, we have

$$d=\frac{\pm(-ab-ab-ab)}{\sqrt{a^2+b^2}}=\frac{\mp 3ab}{\sqrt{a^2+b^2}}.$$

19. From the point (a, b) to the line $ax+by=0$.

Here $Ax+By=C$ is $ax+by=0$, and (x_1, y_1) is (a, b).

Hence, $$d=\frac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}.$$

20. From the point (h, k) to the line $Ax + By + C = D$.

Here (x_1, y_1) is (h, k), and $Ax + By - C$ is $\pm [Ax + By - (D - C)]$, the + or − sign being used according as $D - C$ is positive or negative.

Hence,
$$d = \pm \frac{Ah + Bk - (D - C)}{\sqrt{A^2 + B^2}},$$
in which the upper or lower sign is to be used according as $D - C$ is positive or negative.

21. Find the distance between the two parallels
$$3x + 4y + 15 = 0 \text{ and } 3x + 4y + 5 = 0.$$

Reducing these equations to the normal form, we have
$$-\tfrac{3}{5}x - \tfrac{4}{5}y = 3. \quad (1)$$
$$-\tfrac{3}{5}x - \tfrac{4}{5}y = 1. \quad (2)$$

Since their distances from the origin are 3 and 1 respectively, and they both cross the third quadrant, the distance between them must be 2.

22. Find the distance between the two parallels
$$3x + 4y + 15 = 0 \text{ and } 3x + 4y - 5 = 0.$$

Reducing these equations to the normal form, we have
$$-\tfrac{3}{5}x - \tfrac{4}{5}y = 3. \quad (1)$$
$$\tfrac{3}{5}x + \tfrac{4}{5}y = 1. \quad (2)$$

Since their distances from the origin are 3 and 1 respectively, and (1) crosses the third and (2) the first quadrant, the distance between them must be 4.

23. Find the distance between the two parallels
$$Ax + By = C \text{ and } Ax + By = C'. \quad (C \text{ and } C' \text{ are } +.)$$

Reducing these equations to the normal form, we have
$$\frac{Ax + By}{\sqrt{A^2 + B^2}} = \frac{C}{\sqrt{A^2 + B^2}}, \quad \frac{Ax + By}{\sqrt{A^2 + B^2}} = \frac{C'}{\sqrt{A^2 + B^2}}.$$

These two lines evidently lie on the same side of the centre; hence the distance between them is $\pm \frac{C - C'}{\sqrt{A^2 + B^2}}$, according as $C - C'$ is + or −.

24. Find the distance between the two parallels

$$Ax + By = C \text{ and } -Ax - By = C'.$$

Here the lines lie on opposite sides of the origin; hence the distance between them is $\dfrac{C + C'}{\sqrt{A^2 + B^2}}$.

25. Find the distance between the two parallels

$$y = 5x - 7 \text{ and } y = 5x + 3.$$

Reducing these equations to the normal form, we have

$$\frac{5x}{\sqrt{26}} - \frac{y}{\sqrt{26}} = \frac{7}{\sqrt{26}} \text{ and } -\frac{5x}{\sqrt{26}} + \frac{y}{\sqrt{26}} = \frac{3}{\sqrt{26}}.$$

Since these lines lie on opposite sides of the origin, the distance between them is $\dfrac{10}{\sqrt{26}}$, or $\frac{5}{13}\sqrt{26}$.

26. Find the distance between the two parallels

$$\frac{x}{a} + \frac{y}{b} = 2 \text{ and } \frac{x}{a} + \frac{y}{b} = \frac{1}{2}.$$

Reducing these equations to the normal form, we have

$$\pm \frac{bx + ay}{\sqrt{a^2 + b^2}} = \pm \frac{2ab}{\sqrt{a^2 + b^2}} \text{ and } \pm \frac{bx + ay}{\sqrt{a^2 + b^2}} = \pm \frac{ab}{2\sqrt{a^2 + b^2}},$$

the upper or lower signs being used according as a and b have like or unlike signs.

These lines lie on the same side of the origin; hence the distance between them is

$$\frac{\pm 4ab}{2\sqrt{a^2 + b^2}} - \frac{\pm ab}{2\sqrt{a^2 + b^2}} = \frac{\pm 3ab}{2\sqrt{a^2 + b^2}},$$

in which the upper or lower sign is to be used according as a and b have like or unlike signs.

27. Show that the locus of a point which is equidistant from the lines $3x + 4y - 12 = 0$ and $4x + 3y - 24 = 0$ consists of two straight lines. Find their equations, and draw a figure representing the four lines.

Let (x, y) be the point; then by [12] the distance of the point (x, y) from the line $3x + 4y = 12$ is

$$\frac{3x + 4y - 12}{5},$$

and its distance from the line $4x + 3y = 24$ is

$$\frac{4x + 3y - 24}{5}.$$

First let (x, y) be in the angle between the lines which includes the origin, or its vertical angle; then the distances of (x, y) from the lines will have the same sign, and we have

$$\frac{3x + 4y - 12}{5} = \frac{4x + 3y - 24}{5}, \text{ or } y = x - 12. \tag{1}$$

If (x, y) is in either of the other angles, its distances from the lines will have opposite signs; hence we have

$$\frac{3x + 4y - 12}{5} = -\frac{4x + 3y - 24}{5}, \text{ or } y = -x + \tfrac{36}{7}. \tag{2}$$

Equations (1) and (2), being of the first degree, represent straight lines; and the locus of the point is the bisector of the angles between the given lines, whose equations are (1) and (2). The original equations, as also (1) and (2), are easily constructed.

28. Show that the locus of a point which so moves that the sum of its distances from two given straight lines is constant is a straight line.

Let the point be (x, y), and the straight lines

$$Ax + By = C, \tag{1}$$

and

$$A'x + B'y = C. \tag{2}$$

Now by [12] the distances of (x, y) from (1) and (2) are respectively

$$\frac{Ax + By - C}{\sqrt{A^2 + B^2}} \text{ and } \frac{A'x + B'y - C'}{\sqrt{A'^2 + B'^2}}.$$

Putting the sum of these distances equal to the constant D, we obtain as the equation of the locus

$$\frac{Ax + By - C}{\sqrt{A^2 + B^2}} + \frac{A'x + B'x - C'}{\sqrt{A'^2 + B'^2}} = D. \tag{3}$$

Since (3) is of the first degree, its locus is a straight line.

Ex. 16. Page 54.

1. Find the area of the triangle whose vertices are the points (0, 0), (1, 2), (2, 1).

If (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the vertices of any triangle; then numerically

$$\text{Area} = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \qquad [13]$$

Here (x_1, y_1) is (0, 0), (x_2, y_2) is (1, 2), and (x_3, y_3) is (2, 1). Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[0(-1) + 1(1 - 0) + 2(-2)] = 1\tfrac{1}{2}.$$

2. Find the area of the triangle whose vertices are the points (3, 4), (– 3, – 4), (0, 4).

Here (x_1, y_1) is (3, 4), (x_2, y_2) is (– 3, – 4), and (x_3, y_3) is (0, 4). Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[3(-4-4) - 3(4-4) + 0(4+4)] = 12.$$

3. Find the area of the triangle whose vertices are the points (2, 3), (4, – 5), (– 3, – 6).

Here (x_1, y_1) is (2, 3), (x_2, y_2) is (4, – 5), and (x_3, y_3) is (– 3, – 6). Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[2(-5+6) + 4(-6-3) - 3(3+5)] = 29.$$

4. Find the area of the triangle whose vertices are the points (8, 3), (– 2, 3), (4, – 5).

Here (x_1, y_1) is (8, 3), (x_2, y_2) is (– 2, 3), and (x_3, y_3) is (4, – 5). Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[8(3+5) - 2(-5-3) + 4(3-3)] = 40.$$

5. Find the area of the triangle whose vertices are the points $(a, 0)$, $(-a, 0)$, $(0, b)$.

Here (x_1, y_1) is $(a, 0)$, (x_2, y_2) is $(-a, 0)$, and (x_3, y_3) is $(0, b)$. Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[a(-b) - ab] = ab.$$

6. Compare the formula for the area of a triangle with the result obtained by solving No. 68, p. 42. What, then, is the geometric meaning of that result?

That result regarded as a particular case of the formula for the area of a triangle means that the area of a triangle vanishes when its vertices fall in the same right line.

7. Find the area of the figure having for vertices the points (3, 5), (7, 11), (9, 1).

We give below the solution according to the second method.

By [1] the length of the side whose ends are (3, 5) and (7, 11) is $\sqrt{(7-3)^2+(11-5)^2}$, or base $= \sqrt{52}$.

By [4] the equation of this side is

$$\frac{y-5}{x-3}=\frac{11-5}{7-3}, \text{ or } 2y-3x-1=0.$$

By [12], the distance from (9, 1) to $2y-3x=1$ is

$$\frac{2-27-1}{\sqrt{13}}, \text{ or altitude} = 2\sqrt{13}.$$

Hence, $\quad$ area $= \sqrt{13} \times \sqrt{52} = 26$.

8. Find the area of the figure having for vertices the points (3, −2), (5, 4), (−7, 3).

By [1] the length of the side whose ends are (3, −2) and (5, 4) is $\sqrt{(5-3)^2+(4+2)^2}$, or base $= 2\sqrt{10}$.

By [4] the equation of this side is

$$\frac{y+2}{x-3}=\frac{6}{2}, \text{ or } 3x-y=11.$$

By [12] the distance from (−7, 3) to $3x-y=11$ is

$$\frac{-21-3-11}{\sqrt{10}}, \text{ or altitude} = \tfrac{35}{10}\sqrt{10}.$$

Hence, $\quad$ area $= \sqrt{10} \times \frac{35}{10}\sqrt{10} = 35$.

9. Find the area of the figure having for vertices the points (−1, 2), (4, 4), (6, −3).

By [1] the length of the side whose ends are (−1, 2) and (4, 4) is $\sqrt{5^2+2^2}$, or base $= \sqrt{29}$.

By [4] the equation of this side is

$$\frac{y-2}{x+1}=\frac{2}{5}, \text{ or } 5y-2x=12.$$

By [12] the distance from $(6, -3)$ to $5y - 2x = 12$ is

$$\frac{-15 - 12 - 12}{\sqrt{29}}, \text{ or altitude} = \tfrac{39}{29}\sqrt{29}.$$

Hence, $\quad \text{area} = \tfrac{1}{2}\sqrt{29} \times \tfrac{39}{29}\sqrt{29} = 19\tfrac{1}{2}.$

10. Find the area of the figure having for vertices the points $(0, 0)$, (x_1, y_1), (x_2, y_2).

By [1] the length of the side whose ends are $(0, 0)$ and (x_1, y_1) is $\sqrt{x_1^2 + y_1^2}$, or base $= \sqrt{x_1^2 + y_1^2}$.

B [4] the equation of this side is

$$\frac{y}{x} = \frac{y_1}{x_1}, \text{ or } y_1x - x_1y = 0.$$

By [12] the distance from (x_2, y_2) to $y_1x - x_1y = 0$ is

$$\frac{y_1x_2 - x_1y_2}{\sqrt{x_1^2 + y_1^2}}, \text{ the altitude.}$$

Hence, $\quad \text{area} = \tfrac{1}{2}(y_1x_2 - x_1y_2).$

11. Find the area of the figure having for vertices the points $(2, -5)$, $(2, 8)$, $(-2, -5)$.

Here (x_1, y_1) is $(2, -5)$, (x_2, y_2) is $(2, 8)$, and (x_3, y_3) is $(-2, -5)$. Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[2(8 + 5) - 2(-5 - 8)] = 26.$$

12. Find the area of the figure having for vertices the points $(10, 5)$ $(-2, 5)$, $(-5, -3)$, $(7, -3)$.

By inspection we see that the vertices of this quadrilateral are arranged in order. We have then to find the areas of the two triangles whose vertices are $(10, 5)$, $(-2, 5)$, $(-5, -3)$, and $(-5, -3)$, $(7, -3)$, $(10, 5)$ respectively. By [13] we obtain

$$\text{Area} = \tfrac{1}{2}[10(8) - 2(-8) - 5(0)] = 48,$$

and $$\text{Area} = \tfrac{1}{2}[-5(-8) + 7(8) - 10(0)] = 48.$$

Hence area of quadrilateral $= 96$.

13. Find the area of the figure having for vertices the points $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$.

Here the vertices are arranged in order, and we have to find the areas of the two triangles whose vertices are (0, 0), (5, 0), (9, 11), and (9, 11), (0, 3), (0, 0) respectively. By [13] we obtain

$$\text{Area} = \tfrac{1}{2}[5(11) + 9(0)] = 27\tfrac{1}{2},$$

and $$\text{Area} = \tfrac{1}{2}[9(3)] = 13\tfrac{1}{2}.$$

Hence area of figure = 41.

14. Find the area of the figure having for vertices the points $(a, 1)$, $(0, b)$, $(c, 1)$.

Here (x_1, y_1) is $(a, 1)$, (x_2, y_2) is $(0, b)$, and (x_3, y_3) is $(c, 1)$. Hence, by [13], we obtain

$$\text{Area} = \tfrac{1}{2}[a(b-1) + c(1-b)] = \tfrac{1}{2}(a-c)(b-1).$$

15. Find the area of the figure having for vertices the points (a, b), (b, a), (c, c).

Here (x_1, y_1) is (a, b), (x_2, y_2) is (b, a), and (x_3, y_3) is (c, c). Hence, by [13], we obtain

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}[a(a-c) + b(c-b) + c(b-a)] \\ &= \tfrac{1}{2}(a-b)(a+b-2c).\end{aligned}$$

16. Find the area of the figure having for vertices the points (a, b), (b, a), $(c, -c)$.

Here (x_1, y_1) is (a, b), (x_2, y_2) is (b, a), and (x_3, y_3) is $(c, -c)$. Hence, by [13], we obtain

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}[a(a+c) + b(-c-b) + c(b-a)] \\ &= \tfrac{1}{2}(a^2 - b^2).\end{aligned}$$

17. Find the angles and the area of the triangle whose vertices are $(3, 0)$, $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$.

By [4] the equation of the side through $(3, 0)$ and $(0, 3\sqrt{3})$ is

$$\frac{y}{x-3} = \frac{\cancel{3}\sqrt{3}}{-\cancel{3}}, \text{ or } y = -\sqrt{3}x + 3\sqrt{3}; \tag{1}$$

that of the side through $(0, 3\sqrt{3})$ and $(6, 3\sqrt{3})$ is

$$\frac{y - 3\sqrt{3}}{x} = 0, \text{ or } y = 3\sqrt{3}; \tag{2}$$

and that of the side through $(6, 3\sqrt{3})$ and $(3, 0)$ is

$$\frac{y - 3\sqrt{3}}{x - 6} = \frac{-\cancel{3}\sqrt{3}}{-\cancel{3}}, \text{ or } y = \sqrt{3}x - 3\sqrt{3}. \tag{3}$$

To find the angle between sides (1) and (2), we have

$$\tan\phi = \frac{m - m'}{1 + mm'} = \pm\sqrt{3}, \quad \therefore\ \phi = 60^\circ \text{ or } 120^\circ.$$

To find the angle between sides (2) and (3), we have

$$\tan\phi = \frac{m - m'}{1 + mm'} = \pm\sqrt{3}, \quad \therefore\ \phi = 60^\circ \text{ or } 120^\circ.$$

To find the angle between sides (3) and (1), we have

$$\tan\phi = \frac{m - m'}{1 + mm'} = \pm\sqrt{3}, \quad \therefore\ \phi = 60^\circ \text{ or } 120^\circ.$$

Hence each of these angles is 60°.

By [13] we obtain

$$\text{Area} = \tfrac{1}{2}\,[6\,(-3\sqrt{3})] = 9\sqrt{3}.$$

18. What is the area contained by the lines

$$x = 0,\ \ y = 0,\ \ 5x + 4y = 20\,?$$

The figure is evidently a right triangle whose sides are the intercepts of $5x + 4y = 20$, which are 4 and 5 respectively; hence the area is 10.

19. What is the area contained by the lines

$$x + y = 1,\ \ x - y = 0,\ \ y = 0\,?$$

The intersection of $x + y = 1$ and $x - y = 0$ is $(\frac{1}{2}, \frac{1}{2})$; that of $x - y = 0$ and $y = 0$ is $(0, 0)$; and that of $y = 0$ and $x + y = 1$ is $(1, 0)$.

Now by [13], for the area of the triangle whose vertices are $(\frac{1}{2}, \frac{1}{2})$, $(0, 0)$, and $(1, 0)$ we have

$$\text{Area} = \tfrac{1}{2}\,[\tfrac{1}{2}] = \tfrac{1}{4}.$$

20. What is the area contained by the lines

$$x + 2y = 5,\ \ 2x + y = 7,\ \ y = x + 1\,?$$

The intersection of $x + 2y = 5$ and $2x + y = 7$ is $(3, 1)$; that of $2x + y = 7$ and $y = x + 1$ is $(2, 3)$; and that of $y = x + 1$ and $x + 2y = 5$ is $(1, 2)$.

By [13], for the triangle whose vertices are $(3, 1)$, $(2, 3)$, and $(1, 2)$ we obtain

$$\text{Area} = \tfrac{1}{2}\,[3 + 2 - 2] = 1\tfrac{1}{2}.$$

21. What is the area contained by the lines

$$x + y = 0, \quad x = y, \quad y = 3a?$$

The intersection of $x + y = 0$ and $x = y$ is $(0, 0)$; that of $x = y$ and $y = 3a$ is $(3a, 3a)$; and that of $y = 3a$ and $x + y = 0$ is $(-3a, 3a)$.

By [13], for the triangle whose vertices are $(0, 0)$, $(3a, 3a)$, and $(-3a, 3a)$ we obtain

$$\text{Area} = \tfrac{1}{2}[3a(3a) - 3a(-3a)] = 9a^2.$$

22. What is the area contained by the lines

$$y = 3x, \quad y = 7x, \quad y = c?$$

The intersection of $y = 3x$ and $y = 7x$ is $(0, 0)$; that of $y = 7x$ and $y = c$ is $(\tfrac{1}{7}c, c)$; and that of $y = c$ and $y = 3x$ is $(\tfrac{1}{3}c, c)$.

By [13], for the triangle whose vertices are $(0, 0)$, $(\tfrac{1}{7}c, c)$, and $(\tfrac{1}{3}c, c)$ we obtain

$$\text{Area} = \tfrac{1}{2}[\tfrac{1}{7}c^2 - \tfrac{1}{3}c^2] = \tfrac{2}{21}c^2.$$

23. What is the area contained by the lines
$x = 0, \quad y = 0, \quad x - 4 = 0, \quad y + 6 = 0$?

The figure is evidently a rectangle whose sides are 4 and 6; hence the area $= 24$.

24. What is the area contained by the lines

$$3x + y + 4 = 0, \quad 3x - 5y + 34 = 0, \quad 3x - 2y + 1 = 0?$$

The intersection of $3x + y + 4 = 0$ and $3x - 5y + 34 = 0$ is $(-3, 5)$; that of $3x - 5y + 34 = 0$ and $3x - 2y + 1 = 0$ is $(7, 11)$; and that of $3x - 2y + 1 = 0$ and $3x + y + 4 = 0$ is $(-1, -1)$.

By [13], for the triangle whose vertices are $(-3, 5)$, $(7, 11)$, and $(-1, -1)$, we obtain

$$\text{Area} = \tfrac{1}{2}[-3(12) + 7(-6) - 1(-6)] = 36.$$

25. What is the area contained by the lines

$$x - 5y + 13 = 0, \quad 5x + 7y + 1 = 0, \quad 3x + y - 9 = 0?$$

The intersection of $x - 5y + 13 = 0$ and $5x + 7y + 1 = 0$ is $(-3, 2)$; that of $5x + 7y + 1 = 0$ and $3x + y - 9 = 0$ is $(4, -3)$; and that of $3x + y - 9 = 0$ and $x - 5y + 13 = 0$ is $(2, 3)$.

By [13], for the triangle whose vertices are $(-3, 2)$, $(4, -3)$, and $(2, 3)$, we obtain

$$\text{Area} = \tfrac{1}{2}[-3(-3 - 3) + 4 + 2(5)] = 16.$$

26. What is the area contained by the lines

$$x - y = 0,\quad x + y = 0,\quad x - y = a,\quad x + y = b?$$

The sides $y = x$ and $y = x - a$ are parallel, as also are the sides $y = -x$ and $y = -x + b$; hence the figure is a parallelogram. Therefore its area is twice the area of a triangle whose vertices are any three vertices of the parallelogram.

The intersection of $y = x$ and $y = -x$ is $(0, 0)$; that of $y = -x$ and $y = x - a$ is $(\frac{1}{2}a, -\frac{1}{2}a)$; and that of $y = x - a$ and $y = -x + b$ is

$$\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

By [13], for twice the triangle whose vertices are $(0, 0)$, $(\frac{1}{2}a, -\frac{1}{2}a)$, and $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$ we obtain

$$\text{Area} = \left[\tfrac{1}{2}a\left(\frac{b-a}{2}\right) + \frac{a+b}{2}\left(\tfrac{1}{2}a\right)\right] = \tfrac{1}{2}ab.$$

27. Find the area contained by the lines $x = 0$, $y = 0$, $y = mx + b$.

Since $x = 0$ is the axis of y, and $y = 0$ is the axis of x, the required area is a right triangle whose sides are the intercepts of $y = mx + b$. These intercepts are b and $-(b \div m)$.

Hence, $\quad \text{area} = \frac{1}{2}b(b \div m) = b^2 \div 2m$

28. Find the area contained by the lines $x = 0$, $y = 0$, $\frac{x}{a} + \frac{y}{b} = 1$.

Here the area is a right triangle whose sides are the intercepts of $\frac{x}{a} + \frac{y}{b} = 1$, or a and b.

Hence, $\quad \text{area} = \frac{1}{2}ab$.

29. Find the area contained by the lines $x = 0$, $y = 0$, $Ax + By + C = 0$.

Here the area is a right triangle whose sides are the intercepts of $Ax + By + C = 0$, or $-(C \div A)$ and $-(C \div B)$.

Hence, $\quad \text{area} = \frac{1}{2}(C \div A)(C \div B) = C^2 \div 2AB$.

30. Find the area contained by the lines $y = 3x - 9$, $y = 3x + 5$, $2y = x - 6$, $2y = x + 14$.

Here the area is that of a parallelogram, two of whose parallel sides are $y = 3x - 9$ and $y = 3x + 5$, and the other two are $y = \frac{1}{2}x - 3$ and $y = \frac{1}{2}x + 7$. Hence the area is twice that of a triangle whose vertices are any three vertices of the parallelogram.

The intersection of $y=\frac{1}{2}x-3$ and $y=3x-9$ is $(\frac{12}{5}, -\frac{9}{5})$; that of $y=\frac{1}{2}x-3$ and $y=3x+5$ is $(-\frac{16}{5}, -\frac{23}{5})$; and that of $y=\frac{1}{2}x+7$ with $y=3x-9$ is $(\frac{32}{5}, \frac{51}{5})$.

By [13] for the triangle whose vertices are $(\frac{12}{5}, -\frac{9}{5})$, $(-\frac{16}{5}, -\frac{23}{5})$, and $(\frac{32}{5}, \frac{51}{5})$, we obtain

$$\text{Area} = \tfrac{1}{2}\left[\tfrac{12}{5}(-\tfrac{23}{5}-\tfrac{51}{5}) - \tfrac{16}{5}(\tfrac{51}{5}+\tfrac{9}{5}) + \tfrac{32}{5}(-\tfrac{9}{5}+\tfrac{23}{5})\right] = 28.$$

Hence, area of parallelogram = 56.

31. What is the area of the triangle formed by drawing straight lines from the point (2, 11) to the points in the line $y=5x-6$ for which $x_1=4$, $x_2=7$?

For $x=4$ in $y=5x-6$, $y=14$; hence, (x_1, y_1) is (4, 14).

For $x=7$ in $y=5x-6$, $y=29$; hence, (x_2, y_2) is (7, 29).

By [13] for the triangle whose vertices are (4, 14), (7, 29), and (2, 11) we have

$$\text{Area} = \tfrac{1}{2}[4(29-11)+7(11-14)+2(14-29)] = 10\tfrac{1}{2}.$$

Ex. 17. Page 56.

1. Deduce equation [7], p. 39, from equation [6]

In [6] b = the intercept on the axis of y.

For $y=0$ in [6] $x=-\frac{b}{m}$; hence a, or the intercept on the axis of x equals $-(b \div m)$. Writing [6] in the form

$$\frac{y}{b}+\frac{x}{-(b \div m)}=1,$$

and substituting a for its equal $-(b \div m)$, we have

$$\frac{y}{b}+\frac{x}{a}=1.$$

2. The equation $y=mx+b$ is not so general as the equation $Ax+By+C=0$, because it cannot represent a line parallel to the axis of y. Explain more fully.

The simplest form of the equation of a line parallel to the axis of y is $x-a=0$. Now for $B=0$, $Ax+By+C=0$ assumes this form; but $y=mx+b$ cannot assume this form directly, since the coefficient of y is the absolute constant 1. To make $y=mx+b$ include this form we might write it in the form $y=m(x+c)$ in which $mc=b$

When the line is parallel to the axis of y, $m = \infty$, and the equation $y = m(x + c)$ is satisfied only for $x + c = 0$, which represents a line parallel to the axis of y.

3. Determine for the line $x = 2$ the values of a, b, γ, p, and α.

Reducing $Ax + By = C$ to the three forms [6], [7], [8], we have

$$y = -(A \div B)\,x + (C \div B), \tag{1}$$

$$\frac{x}{C \div A} + \frac{y}{C \div B} = 1, \tag{2}$$

$$\frac{Ax}{\sqrt{A^2 + B^2}} + \frac{By}{\sqrt{A^2 + B^2}} = \frac{C}{\sqrt{A^2 + B^2}}. \tag{3}$$

Hence,

$$\left.\begin{aligned} &a = C \div A;\ b = C \div B;\ \tan\gamma = -(A \div B);\\ &p = \frac{C}{\sqrt{A^2 + B^2}};\ \cos\alpha = \frac{A}{\sqrt{A^2 + B^2}};\ \sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}. \end{aligned}\right\} \tag{4}$$

Here $A = 1$, $B = 0$, $C = 2$. Substituting these values in (4), we have $a = 2$; $b = 2 \div 0 = \infty$; $\tan\gamma = -\infty$ or $\gamma = 90^\circ$; $p = 2 \div 1 = 2$; $\cos\alpha = 1$, $\sin\alpha = 0$, or $\alpha = 0^\circ$.

4. Determine for the line $x = y$ the values of a, b, γ, p, and α.

Here $A = -1$, $B = +1$, $C = 0$. Substituting these values in (4) of No. 3, we have

$$a = 0;\ b = 0;\ \tan\gamma = 1,\ \gamma = 45^\circ;\ p = 0;\ \cos\alpha = -\tfrac{1}{2}\sqrt{2},$$
$$\sin\alpha = +\tfrac{1}{2}\sqrt{2},\ \text{or}\ \alpha = 135^\circ.$$

5. Determine for the line $y + 1 = \sqrt{3}\,(x + 2)$ the values of a, b, γ, p, and α.

Putting this equation in the form of $Ax + By = C$, we have

$$-\sqrt{3}\,x + y = 2\sqrt{3} - 1.$$

Here $A = -\sqrt{3}$, $B = 1$, $C = 2\sqrt{3} - 1$. Substituting these values in (4) of No. 3, we have

$$a = \frac{2\sqrt{3} - 1}{-\sqrt{3}} = \tfrac{1}{3}\sqrt{3} - 2;\ b = 2\sqrt{3} - 1;\ \tan\gamma = \sqrt{3},\ \text{or}\ \gamma = 60^\circ;$$

$$p = \frac{2\sqrt{3} - 1}{\sqrt{3} + 1} = \sqrt{3} - \tfrac{1}{2};\ \cos\alpha = -\tfrac{1}{2}\sqrt{3},\ \sin\alpha = \tfrac{1}{2},\ \text{or}\ \alpha = 150^\circ.$$

6. Determine for the line $x + \sqrt{3}y = 2$ the values of a, b, γ, p, and α.

Here $A = 1$, $B = \sqrt{3}$, $C = 2$. Substituting these values in (4) of No. 3, we have

$$a = 2;\ b = \tfrac{2}{3}\sqrt{3};\ \tan\gamma = -\tfrac{1}{3}\sqrt{3},\ \text{or}\ \gamma = 150^\circ;\ p = 1;$$

$$\cos\alpha = \tfrac{1}{2},\ \sin\alpha = \tfrac{1}{2}\sqrt{3},\ \text{or}\ \alpha = 60^\circ.$$

7. Determine for the line $x - \sqrt{3}y = 2$ the values of a, b, γ, p, and α.

Here $A = 1$, $B = -\sqrt{3}$, $C = 2$. Substituting these values in (4) of No. 3, we have

$$a = 2;\ b = -\tfrac{2}{3}\sqrt{3};\ \tan\gamma = \tfrac{1}{3}\sqrt{3},\ \text{or}\ \gamma = 30^\circ;\ p = 1;$$

$$\cos\alpha = \tfrac{1}{2},\ \sin\alpha = -\tfrac{1}{2}\sqrt{3},\ \text{or}\ \alpha = 300^\circ.$$

8. Determine for the line $\sqrt{3}x - y = 2$ the values of a, b, γ, p, and α.

Here $A = \sqrt{3}$, $B = -1$, $C = 2$. Substituting these values in (4) of No. 3, we have

$$a = \tfrac{2}{3}\sqrt{3};\ b = -2;\ \tan\gamma = \sqrt{3},\ \text{or}\ \gamma = 60^\circ;\ p = 1;$$

$$\cos\alpha = \tfrac{1}{2}\sqrt{3},\ \sin\alpha = -\tfrac{1}{2},\ \text{or}\ \alpha = 330^\circ.$$

9. Find the equations of the diagonals of the figure formed by the lines $3x - y + 9 = 0$, $3x = y - 1$, $5x + 3y = 18$, $5x + 3y = 2$. What kind of quadrilateral is it? Why?

Writing the equations of the lines in the form $y = mx + b$, we have

$$y = 3x + 9. \qquad (1) \qquad\qquad y = -\tfrac{5}{3}x + 6. \qquad (3)$$

$$y = 3x + 1. \qquad (2) \qquad\qquad y = -\tfrac{5}{3}x + \tfrac{2}{3}. \qquad (4)$$

Lines (1) and (2) are parallel, since the slope of each is 3.
Lines (3) and (4) are also parallel, since the slope of each is $-\frac{5}{3}$.
Since the opposite sides are parallel, the figure is a parallelogram.
Solving (1) and (3) as simultaneous, we obtain the vertex $(-\frac{9}{14}, \frac{99}{14})$.
Solving (1) and (4) as simultaneous, we obtain the vertex $(-\frac{25}{14}, \frac{51}{14})$.
Solving (2) and (3) as simultaneous, we obtain the vertex $(\frac{15}{14}, \frac{59}{14})$.
Solving (2) and (4) as simultaneous, we obtain the vertex $(-\frac{1}{14}, \frac{11}{14})$.
Of these vertices the first and last are opposite, since the first is the

vertex of the angle between the sides (1) and (3), and the last, of the angle between (2) and (4). The diagonal through $(-\frac{9}{14}, \frac{99}{14})$ and $(-\frac{1}{14}, \frac{11}{14})$ is

$$\frac{y - \frac{99}{14}}{x + \frac{9}{14}} = \frac{\frac{11}{14} - \frac{99}{14}}{-\frac{1}{14} + \frac{9}{14}}, \text{ or } 11x + y = 0.$$

The diagonal through the other two vertices $(-\frac{25}{14}, \frac{51}{14})$ and $(\frac{15}{14}, \frac{59}{14})$ is

$$\frac{y - \frac{51}{14}}{x + \frac{25}{14}} = \frac{\frac{59}{14} - \frac{51}{14}}{\frac{15}{14} + \frac{25}{14}}, \text{ or } x - 5y + 20 = 0.$$

10. Find the distance between the parallels $9x = y + 1$ and $9x = y - 7$.

Writing these equations in the normal form, we have

$$\frac{9}{\sqrt{82}}x - \frac{1}{\sqrt{82}}y = \frac{1}{\sqrt{82}}, \tag{1}$$

and

$$-\frac{9}{\sqrt{82}}x + \frac{1}{\sqrt{82}}y = \frac{7}{\sqrt{82}}. \tag{2}$$

From their signs we see that (1) and (2) lie on opposite sides of the origin, (1) crossing the fourth quadrant, and (2) crossing the second. Hence the distance between them is equal to the sum of their distances from the origin; that is,

$$\frac{8}{\sqrt{82}}, \text{ or } \frac{4}{41}\sqrt{82}.$$

11. The vertices of a quadrilateral are (3, 12), (7, 9), (2, -3), (-2, 0). Find the equations of its sides and its area.

The vertices in order are (3, 12), (7, 9), (2, -3), (-2, 0).

The line through (3, 12) and (7, 9) by [4] is

$$\frac{y - 12}{x - 3} = \frac{9 - 12}{7 - 3} = -\frac{3}{4}, \text{ or } 3x + 4y = 57. \tag{1}$$

The line through (7, 9) and (2, -3) is

$$\frac{y - 9}{x - 7} = \frac{-3 - 9}{2 - 7} = \frac{12}{5}, \text{ or } 12x - 5y = 39. \tag{2}$$

The line through (2, -3) and (-2, 0) is

$$\frac{y + 3}{x - 2} = \frac{3}{-2 - 2} = -\frac{3}{4}, \text{ or } 3x + 4y + 6 = 0. \tag{3}$$

The line through $(-2, 0)$ and $(3, 12)$ is

$$\frac{y}{x+2}=\frac{12}{3+2}=\frac{12}{5},\ \text{or}\ 12x-5y+24=0. \qquad (4)$$

Sides (1) and (3) are parallel, as are also (2) and (4); hence the figure is a parallelogram.

The length of the side between the vertices $(3, 12)$ and $(7, 9)$ is $\sqrt{(7-3)^2+(9-12)^2}$, or 5; and the distance between this side, whose equation is (1), and its opposite side (3) is $\frac{57}{5}+\frac{6}{5}$, or $\frac{63}{5}$. Hence the area is 63.

Lines (1) and (3) lie on opposite sides of the origin, and their distances from it are obtained by reducing their equations to the normal form.

12. The vertices of a quadrilateral are $(6, -4)$, $(4, 4)$, $(-4, 2)$, $(-8, -6)$. Prove that the lines joining the middle points of adjacent sides form a parallelogram. Find the area of this parallelogram.

The vertices in order are $(6, -4)$, $(4, 4)$, $(-4, 2)$, $(-8, -6)$.

The mid-point between $(6, -4)$ and $(4, 4)$ is $(5, 0)$; that between $(4, 4)$ and $(-4, 2)$ is $(0, 3)$; that between $(-4, 2)$ and $(-8, -6)$ is $(-6, -2)$; that between $(-8, -6)$ and $(6, -4)$ is $(-1, -5)$.

The line through the mid-points $(5, 0)$ and $(0, 3)$ is

$$\frac{y}{x-5}=\frac{3}{-5},\ \text{or}\ 3x+5y=15. \qquad (1)$$

The line through the mid-points $(0, 3)$ and $(-6, -2)$ is

$$\frac{y-3}{x}=\frac{-2-3}{-6}=\frac{5}{6},\ \text{or}\ 5x-6y+18=0. \qquad (2)$$

The line through the mid-points $(-6, -2)$ and $(-1, -5)$ is

$$\frac{y+2}{x+6}=\frac{-5+2}{-1+6}=-\frac{3}{5},\ \text{or}\ 3x+5y+28=0. \qquad (3)$$

The line through the mid-points $(-1, -5)$ and $(5, 0)$ is

$$\frac{y+5}{x+1}=\frac{5}{5+1}=\frac{5}{6},\ \text{or}\ 5x-6y=25. \qquad (4)$$

Lines (1) and (3) are parallel, as are also (2) and (4); hence the figure is a parallelogram.

The length of the side between the vertices $(5, 0)$ and $(0, 3)$ is $\sqrt{25+9}$, or $\sqrt{34}$; and the distance between this side and its opposite side (3) is $\frac{15}{\sqrt{34}}+\frac{28}{\sqrt{34}}$, or $\frac{43}{\sqrt{34}}$; hence the area is 43. Lines (1) and (3) are on opposite sides of the origin.

13. Find the equation of a line passing through (3, 4), and also perpendicular to the axis of x.

The equation of any line perpendicular to the axis of x is $x = a$. If the line passes through (3, 4), x must equal 3; hence $x = 3$ is the equation required.

14. Find the equation of a line passing through (3, 4), and also making the angle 45° with the axis of x.

The line through (x_1, y_1) having slope m is

$$y - y_1 = m(x - x_1). \tag{5}$$

Here (x_1, y_1) is (3, 4) and $m = \tan\gamma = \tan 45^\circ = 1$. Substituting these values in [5], we have

$$y - 4 = 1(x - 3) \text{ or } y = x + 1.$$

If the line makes an angle of 45° with the negative direction of the axis of x, we have $m = \tan\gamma = \tan 135^\circ = -1$; and $y - 4 = -1(x - 3)$, or $y = -x + 7$.

15. Find the equation of a line passing through (3, 4), and also parallel to the line $5x + 6y + 8 = 0$.

Here (x_1, y_1) is (3, 4), and $m = -\frac{5}{6}$. Substituting in [5], we have

$$y - 4 = -\tfrac{5}{6}(x - 3), \text{ or } 5x + 6y - 39 = 0,$$

as the required equation.

16. Find the equation of a line passing through (3, 4), and also intercepting on the axis of y the distance -10.

If in $y = mx + b$, $b = -10$, and the line passes through (3, 4), we have $4 = 3m - 10$, or $m = \frac{14}{3}$. Hence the required equation is $y = \frac{14}{3}x - 10$, or $14x - 3y - 30 = 0$.

17. Find the equation of a line passing through (3, 4), and also passing through the point half way between (1, −4) and (−5, 4).

The point midway between (1, −4) and (−5, 4) is (−2, 0).
The line through (3, 4) and (−2, 0) is, by [4],

$$\frac{y - 4}{x - 3} = \frac{-4}{-2 - 3} = \frac{4}{5}, \text{ or } 4x - 5y + 8 = 0.$$

18. Find the equation of a line passing through (3, 4), and also perpendicular to the line joining (3, 4) and (– 1, 0).

The line through (3, 4) and (– 1, 0) is, by [4],

$$\frac{y-4}{x-3}=\frac{-4}{-1-3}=1, \text{ or } x-y+1=0. \qquad (1)$$

The slope of (1) is 1; hence the line through (3, 4) perpendicular to (1) is $y-4=-(x-3)$ or $x+y-7=0$.

19. Find the equation of a line parallel to the line joining (x_1, y_1) and (x_2, y_2), and passing through (x_3, y_3).

The slope of the line through (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$; hence the line through (x_3, y_3) parallel to this line is $y-y_3=\frac{y_2-y_1}{x_2-x_1}(x-x_3)$.

20. Find the equations of the lines passing through (8, 3), (4, 3), (– 5, – 2).

The line through (8, 3) and (4, 3) is

$$\frac{y-3}{x-8}=\frac{3-3}{4-8}=0, \text{ or } y=3.$$

The line through (4, 3) and (– 5, – 2) is

$$\frac{y-3}{x-4}=\frac{-2-3}{-5-4}=\frac{5}{9}, \text{ or } 5x-9y+7=0.$$

The line through (– 5, – 2) and (8, 3) is

$$\frac{y+2}{x+5}=\frac{3+2}{8+5}=\frac{5}{13}, \text{ or } 5x-13y-1=0.$$

21. Find the equation of a line passing through the intersection of the lines $2x+5y+8=0$ and $3x-4y-7=0$, and perpendicular to the latter line.

The line through the intersection of

$$2x+5y+8=0 \text{ and } 3x-4y-7=0 \text{ is}$$

$$2x+5y+8+k(3x-4y-7)=0. \qquad (1)$$

The slope of (1) is $\frac{-(2+3k)}{5-4k}$, and that of $3x-4y-7=0$ is $\frac{3}{4}$.

Hence if these lines are perpendicular, we must have

$$\frac{-(2+3k)}{5-4k}=-\frac{4}{3}, \text{ or } k=\frac{14}{25}.$$

Substituting this value of k in (1), we obtain after reduction

$$92x + 69y + 102 = 0,$$

as the required equation.

22. Find the equation of a line perpendicular to the line $4x - y = 0$, and passing through that point of the given line whose abscissa is 2.

For $x = 2$ in $4x - y = 0$, $y = 8$; and the point is (2, 8).

The slope of $4x - y = 0$ is 4; hence the slope of the required line is $-\frac{1}{4}$. The line through (2, 8), having slope $-\frac{1}{4}$, is

$$y - 8 = -\tfrac{1}{4}(x - 2), \text{ or } x + 4y = 34.$$

23. Find the equation of a line parallel to the line $3x + 4y = 0$, and passing through the intersection of the lines $x - 2y - a = 0$ and $x + 3y - 2a = 0$.

The line through the intersection of the two given lines is

$$x - 2y - a + k(x + 3y - 2a) = 0, \tag{1}$$

whose slope is $\frac{1+k}{2-3k}$. The slope of $3x + 4y = 0$ is $-\frac{3}{4}$.

Putting these slopes equal to each other, we have

$$\frac{1+k}{2-3k} = -\frac{3}{4}, \text{ or } k = 2.$$

Substituting this value of k in (1), we obtain as the required equation

$$3x + 4y - 5a = 0.$$

24. Find the equation of a line through (4, 3), such that the given point bisects the portion contained between the axes.

If the point (4, 3) bisects the portion of the line between the axes, the lines through (4, 3) parallel to the axes must bisect the intercepts of the line on the axes; hence, the intercepts of the line are 8 and 6, and its equation is

$$\frac{x}{8} + \frac{y}{6} = 1, \text{ or } 3x + 4y = 24.$$

25. Find the equation of a line through (x_1, y_1), such that the given point bisects the portion contained between the axes.

If the point (x_1, y_1) bisects the portion of the line between the axes, the intercepts of the line on the axes must be $2x_1$ and $2y_1$ respectively, and its equation is

$$\frac{x}{2x_1}+\frac{y}{2y_1}=1, \text{ or } y-y_1=-\frac{y_1}{x_1}(x-x_1).$$

26. Find the equation of a line through (4, 3), and forming with the axes in the second quadrant a triangle whose area is 8.

The line through (4, 3) is

$$y-3=m(x-4), \text{ or } y=mx+3-4m. \qquad (1)$$

The intercepts of (1) on the axes are $\frac{4m-3}{m}$ and $3-4m$.

When the line crosses the second quadrant, $\frac{4m-3}{m}$ is negative and $3-4m$ is positive. Hence the product of $\frac{3-4m}{m}$ and $3-4m$, or $\frac{(3-4m)^2}{m}$, must equal twice the area of the triangle, or 16.

Solving $\frac{(3-4m)^2}{m}=16$ for m, we find $m=\frac{1}{4}$ and $\frac{9}{4}$, of which values $\frac{1}{4}$ applies to this problem.

Substituting this value in (1), we obtain as the required equation $4y=x+8$.

27. Find the equation of a line through (4, 3), and forming with the axes in the fourth quadrant a triangle whose area is 8.

Here we would have the same expression for the intercepts as in No. 26; but $\frac{4m-3}{m}$ is positive and $3-4m$ is negative; hence we have, as before, $\frac{(4m-3)^2}{m}=16$.

The value $m=\frac{9}{4}$ applies to this example. Substituting this value of m in (1) of No. 26, we have as the required equation $4y=9x-24$.

28. Find the equation of a line through $(-4, 3)$, such that the portion between the axes is divided by the given point in the ratio $5:3$.

Let P (Fig. 39) be the point $(-4, 3)$, and let $X'N$ be the position of the line through it so that $X'P : PN :: 5 : 3$; then $X'N : PN :: 8 : 3$ and $X'N : X'P :: 8 : 5$.

Draw PM parallel to OY, and PS parallel to OX'; then

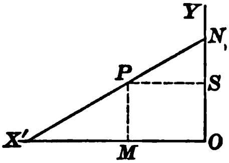

FIG. 39.

$$OX' : OM :: NX' : NP :: 8 : 3,$$

or $$OX' = \tfrac{8}{3}\,OM = -\tfrac{32}{3},$$

and $$ON : OS :: X'N : X'P :: 8 : 5,$$

or $$ON = \tfrac{8}{5}\,OS = \tfrac{24}{5}.$$

Hence the intercepts of NX' on OX' and OY are $-\frac{32}{3}$ and $\frac{24}{5}$, and its equation is

$$\frac{x}{-\frac{32}{3}} + \frac{y}{\frac{24}{5}} = 1, \text{ or } 9x - 20y + 96 = 0.$$

If the line is so situated that $NP : PX' :: 5 : 3$, then

$$NX' : NP :: 8 : 5, \text{ and } NX' : PX' :: 8 : 3.$$
$$OX' : OM :: NX' : NP :: 8 : 5, \text{ or } OX' = \tfrac{8}{5}\,OM = -\tfrac{32}{5}.$$
$$ON : OS :: NX' : PX' :: 8 : 3, \text{ or } ON = \tfrac{8}{3}\,OS = 8.$$

Hence the intercepts are $-\frac{32}{5}$ and 8, and the equation is

$$\frac{x}{-\frac{32}{5}} + \frac{y}{8} = 1, \text{ or } 5x - 4y + 32 = 0.$$

29. Find the equation of a line dividing the distance between $(-3, 7)$ and $(5, -4)$ in the ratio $4 : 7$, and perpendicular to the line joining these points.

If in [3], (x_1, y_1) is $(-3, 7)$, (x_2, y_2) is $(5, -4)$, and $m : n = 4 : 7$, we have

$$x = \frac{20 - 21}{11} = -\frac{1}{11},\quad y = \frac{-16 + 49}{11} = 3.$$

Hence the point of division is $(-\frac{1}{11}, 3)$.

The line through $(-3, 7)$ and $(5, -4)$ is

$$\frac{y - 7}{x + 3} = \frac{-4 - 7}{5 + 3} = -\frac{11}{8}, \text{ or } 11x + 8y = 23,$$

whose slope is $-\frac{11}{8}$. The line through $(-\frac{1}{11}, 3)$ having the slope $\frac{8}{11}$ is

$$y - 3 = \tfrac{8}{11}(x + \tfrac{1}{11}), \text{ or } 88x - 121y + 371 = 0.$$

30. Find the equations of the two lines through (3, 5) making the angle 45° with the line $2x - 3y - 7 = 0$.

The lines through (x_1, y_1) making the angle ϕ with the line $y = mx + b$ is

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m \tan\phi}(x - x_1). \qquad [11]$$

Here (x_1, y_1) is (3, 5), $\phi = 45°$, and $m = \frac{2}{3}$. Making these substitutions in [11], we have

$$y - 5 = \frac{\frac{2}{3} \pm 1}{1 \mp \frac{2}{3}}(x - 3), \text{ or } \begin{cases} 5x - y = 10. \\ x + 5y = 28. \end{cases}$$

31. Find the equations of the two lines through (7, − 5) which make the angle 45° with the line $6x - 2y + 3 = 0$.

Here (x_1, y_1) is (7, − 5), $\tan\phi = \tan 45° = 1$, $m = 3$. Substituting these values in [11], we have

$$y + 5 = \frac{3 \pm 1}{1 \mp 3}(x - 7), \text{ or } \begin{cases} 2x + y = 9. \\ x - 2y = 17. \end{cases}$$

32. Find the equations of the line making the angle 45° with the line joining (7, − 1) and (− 3, 5), and intercepting the distance 5 on the axis of x.

The line through (7, − 1) and (− 3, 5) is

$$\frac{y + 1}{x - 7} = \frac{5 + 1}{-3 - 7} = -\frac{3}{5}, \text{ or } 3x + 5y = 16. \qquad (1)$$

Here (x_1, y_1) is (5, 0), $\tan\phi = \tan 45° = 1$, $m = -\frac{3}{5}$. Substituting these values in [11], we have

$$y = \frac{-\frac{3}{5} \pm 1}{1 \pm \frac{3}{5}}(x - 5), \text{ or } \begin{cases} x - 4y - 5 = 0. \\ 4x + y - 20 = 0. \end{cases}$$

33. Find the equations of the two lines which pass through the origin and trisect the portion of the line $x + y = 1$ included between the axes.

For $x = 0$ in $x + y = 1$, $y = 1$; for $y = 0$, $x = 1$; hence the ends of the portion of $x + y = 1$ included between the axes are (0, 1) and (1, 0).

If in [3] (x_1, y_1) is (0, 1), (x_2, y_2) is (1, 0), and $m : n = 1 : 2$, we have $x = \frac{1}{3}$, $y = \frac{2}{3}$, or one point of trisection is $(\frac{1}{3}, \frac{2}{3})$.

If $m : n :: 2 : 1$, we have $x = \frac{2}{3}$, $y = \frac{1}{3}$, or the second point of trisection is $(\frac{2}{3}, \frac{1}{3})$.

The line through the origin and $(\frac{1}{3}, \frac{2}{3})$ is $y = 2x$; and the line through the origin and $(\frac{2}{3}, \frac{1}{3})$ is $2y = x$.

34. Find the equations of the two lines parallel to the line $4x + 5y + 11 = 0$, at the distance 3 from it.

Writing $4x + 5y + 11 = 0$, in the normal form, we have

$$-\frac{4}{\sqrt{41}}x - \frac{5}{\sqrt{41}}y = \frac{11}{\sqrt{41}}, \tag{1}$$

which evidently crosses the third angle and is at the distance $\frac{11}{\sqrt{41}}$ from the origin.

Now the line crossing the third angle at the distance 3 from (1) is evidently

$$-\frac{4}{\sqrt{41}}x - \frac{5}{\sqrt{41}}y = \frac{11}{\sqrt{41}} + 3, \text{ or } 4x + 5y + 11 + 3\sqrt{41} = 0.$$

The other required line must cross the first angle, and its distance from the origin must be $3 - \frac{11}{\sqrt{41}}$; hence its equation is

$$\frac{4}{\sqrt{41}}x + \frac{5}{\sqrt{41}}y = 3 - \frac{11}{\sqrt{41}}, \text{ or } 4x + 5y + 11 - 3\sqrt{41} = 0.$$

35. Find the equations of the bisectors of the angles contained between the lines $y = 2x + 4$, $-y = 3x + 6$.

There are evidently two bisectors: one bisecting the angle in which the origin lies; the other bisecting the supplementary angle. Now every point in either bisector is equally distant from the sides of the angle. Let (x, y) be any point in the bisector of the angle which includes the origin; then, by § 49, the expressions for its distances from the two sides will be equal numerically, and have like signs; hence we have

$$\frac{y - 2x - 4}{\sqrt{5}} = \frac{-y - 3x - 6}{\sqrt{10}}, \text{ or } y = (7 - 5\sqrt{2})(x + 2) \tag{1}$$

Since (x, y) is any point in this bisector, (1) is its equation.

The expressions for the distances of any point in the other bisector from the sides of the angle will be numerically equal, but unlike in sign; hence the equation of this bisector is

$$\frac{y - 2x - 4}{\sqrt{5}} = -\frac{-y - 3x - 6}{\sqrt{10}}, \text{ or } y = (7 + 5\sqrt{2})(x + 2).$$

36. Find the equations of the bisectors of the angles contained between the lines $2x - 5y = 0$, $4x + 3y = 12$.

Proceeding as in No. 35, we find that the equation of the bisector of the angle in which the origin lies is

$$\frac{2x-5y}{\sqrt{29}} = \frac{4x+3y-12}{5},$$

and the bisector of the supplemental angles is

$$\frac{2x-5y}{\sqrt{29}} = -\frac{4x+3y-12}{5}.$$

37. Find the equations of the two lines which pass through (3, 12), and whose distance from (7, 2) is equal to $\sqrt{58}$.

The lines through (3, 12) are

$$y - 12 = m(x-3), \tag{1}$$

the distances of which from (x_1, y_1), by [12], are

$$\pm\frac{y_1 - 12 - m(x-3)}{\sqrt{1+m^2}}.$$

Here (x_1, y_1) is (7, 2), and the distance is to be $\sqrt{58}$; hence we have

$$\pm\frac{2 - 12 - m(7-3)}{\sqrt{1+m^2}} = \sqrt{58},$$

or

$$(10 - 4m)^2 = 58(1+m^2).$$

Whence $m = \frac{7}{3}$ and $-\frac{3}{7}$. Substituting these values in (1), we obtain as the required equations

$$7x - 3y + 15 = 0,\ 3x + 7y - 93 = 0.$$

38. Find the equations of the two lines which pass through (−2, 5), and are each equidistant from (3, −7) and (−4, 1).

The equation of lines through (−2, 5) is

$$y - 5 = m(x+2), \tag{1}$$

the distance of which from (x_1, y_1) is

$$\pm\frac{y_1 - 5 - m(x_1+2)}{\sqrt{1+m^2}}.$$

If (x_1, y_1) is $(3, -7)$, we have as this distance

$$\pm\frac{-12-5m}{\sqrt{1+m^2}}.$$

If (x_1, y_1) is $(-4, 1)$, we have

$$\pm\frac{-4+2m}{\sqrt{1+m^2}}.$$

That these expressions may be numerically equal, we must have

$$-12-5m=\pm(2m-4);$$

that is, $m=-\frac{8}{7}$ and $-\frac{16}{3}$.

Substituting these values in (1), we obtain as the required equations,

$$8x+7y=19,\quad 16x+3y+17=0.$$

39. Find the angle contained between the lines $y+3=2x$ and $y+3x=2$.

If ϕ be the angle between the lines $y=mx+b$ and $y=m'x+b'$, then

$$\tan\phi=\frac{m-m'}{1+mm'}. \qquad [10]$$

Here $m=2$, and $m'=-3$. Substituting these values in [10], we have $\tan\phi=\frac{2+3}{1-6}=-1$, or $\phi=135^\circ$.

Hence two of the angles formed by these lines are 135° each, and the other two 45° each.

40. Find the angle contained between the lines $y=5x-7$ and $5y+x-3=0$.

Here $m=5$ and $m'=-\frac{1}{5}$. As these slopes are negative reciprocals of each other, the lines are at right angles to each other.

41. Find the distance from the intersection of the lines $3x+2y+4=0$, $2x+5y+8=0$ to the line $y=5x+6$.

Regarding the two given equations as simultaneous, and solving, we find that their point of intersection is $(-\frac{4}{11}, -\frac{16}{11})$. The distance of (x_1, y_1) from $y=5x+6$ is

$$\frac{y_1-5x_1-6}{\sqrt{26}}.$$

If (x_1, y_1) is $(-\frac{4}{11}, -\frac{16}{11})$, we have

$$\frac{-\frac{16}{11}+\frac{20}{11}-6}{\sqrt{26}}, \text{ or } \frac{-62}{11\sqrt{26}}, \text{ or } \frac{-31}{143}\sqrt{26}.$$

Since this distance is —, the origin and point are on the same side of the line.

42. Find the distance from the point (h, k) to the line $\frac{x}{a}+\frac{y}{b}=1$.

By [12] the distance from (x_1, y_1) to $\frac{x}{a}+\frac{y}{b}=1$, or $bx+ay=ab$ is

$$\pm\frac{bx_1+ay_1-ab}{\sqrt{a^2+b^2}}.$$

If (x_1, y_1) is (h, k), we have

$$d=\pm\frac{bh+ak-ab}{\sqrt{a^2+b^2}}.$$

43. Find the distance from the origin to the line $hx+ky=c^2$.

This equation reduced to the normal form is

$$\frac{h}{\sqrt{h^2+k^2}}x+\frac{k}{\sqrt{h^2+k^2}}y=\frac{c^2}{\sqrt{h^2+k^2}}. \qquad (1)$$

Hence the distance of (1) from the origin is $\frac{c^2}{\sqrt{h^2+k^2}}$.

44. Find the distance from the point $(a, 0)$ to the line $y=mx+\frac{a}{m}$.

By [12] the distance from (x_1, y_1) to $y=mx+\frac{a}{m}$ is

$$\pm\frac{y_1-mx_1-(a\div m)}{\sqrt{1+m^2}}.$$

If (x_1, y_1) is $(a, 0)$, we have as the distance

$$\pm\frac{ma-(a\div m)}{\sqrt{1+m^2}}, \text{ or } \pm\frac{a}{m}\sqrt{1+m^2}.$$

45. Find the area included between the following lines:

$$x=y, \quad x+y=0, \quad x=c.$$

The intersection of $x=y$ and $x=c$ is (c, c), and the intersection of $x+y=0$ and $x=c$ is $(c, -c)$. Hence if we regard $x=c$ as the base of the triangle, the altitude is c and the length of the base is $2c$; and therefore the area is c^2.

46. Find the area included between the following lines:

$$x + y = k,\ \ 2x = y + k,\ \ 2y = x + k.$$

The intersection of $x + y = k$ and $2x = y + k$ is $(\frac{2}{3}k, \frac{1}{3}k)$.

The intersection of $2x = y + k$ and $2y = x + k$ is (k, k).

The intersection of $2y = x + k$ and $x + y = k$ is $(\frac{1}{3}k, \frac{2}{3}k)$.

Hence the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) are $(\frac{2}{3}k, \frac{1}{3}k)$, (k, k), $(\frac{1}{3}k, \frac{2}{3}k)$ respectively. Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[\tfrac{2}{3}k(k - \tfrac{2}{3}k) + k(\tfrac{2}{3}k - \tfrac{1}{3}k) + \tfrac{1}{3}k(\tfrac{1}{3}k - k)] = \tfrac{1}{6}k^2.$$

47. Find the area included between the following lines:

$$\frac{x}{a} + \frac{y}{b} = 1,\ \ y = 2x + b,\ \ x = 2y + a.$$

The intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $y = 2x + b$ is $(0, b)$, since they both have the same intercept on the axis of y.

The intersection of $y = 2x + b$ and $x = 2y + a$ is $\left(-\frac{2b + a}{3}, -\frac{b + 2a}{3}\right)$.

The intersection of $x = 2y + a$ and $\frac{x}{a} + \frac{y}{b} = 1$ is $(a, 0)$.

Hence the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) are

$$(0, b),\ \left(-\frac{2b + a}{3}, -\frac{b + 2a}{3}\right),\ (a, 0).$$

Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}\left[-\frac{2b + a}{3}(-b) + a\frac{4b + 2a}{3}\right] = \frac{2a^2 + 5ab + 2b^2}{6}.$$

48. Find the area included between the following lines: $y = 4x + 7$ and the lines which join the origin to those points of the given line whose ordinates are -1 and 19.

For $y = -1$ in $y = 4x + 7$, $x = -2$; for $y = 19$, $x = 3$; hence the points on the given line are $(-2, -1)$ and $(3, 19)$.

The distance between $(-2, -1)$ and $(3, 19)$ is $\sqrt{25 + 400}$ or $\sqrt{425}$.

The distance of $y = 4x + 7$ from the origin is $\frac{7}{\sqrt{17}}$. Hence the base of the triangle is $\sqrt{425}$, and its altitude is $\frac{7}{\sqrt{17}}$; and therefore its area is

$$\frac{7}{2}\frac{\sqrt{425}}{\sqrt{17}} = \frac{35}{2} = 17\tfrac{1}{2}.$$

49. Find the area included by the lines joining the middle points of the sides of the triangle formed by the lines

$$x - 5y + 11 = 0, \quad 11x + 6y - 1 = 0, \quad x + y + 4 = 0.$$

The intersection of $x - 5y + 11 = 0$ and $11x + 6y - 1 = 0$ is $(-1, 2)$.
The intersection of $11x + 6y - 1 = 0$ and $x + y + 4 = 0$ is $(5, -9)$.
The intersection of $x + y + 4 = 0$ and $x - 5y + 11 = 0$ is $(-\frac{31}{6}, \frac{7}{6})$.
Hence, the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) are $(-1, 2)$, $(5, -9)$, $(-\frac{31}{6}, \frac{7}{6})$, respectively. Substituting these values in [13], we obtain

$$\text{Area} = \tfrac{1}{2}[-1(-9 - \tfrac{7}{6}) + 5(\tfrac{7}{6} - 2) - \tfrac{31}{6}(2 + 9)] = 25\tfrac{5}{12}.$$

Hence the required area $= 6\frac{17}{48}$.

50. Find the area of the quadrilateral whose vertices are (0, 0), (0, 5), (11, 9), (7, 0).

The area consists of the two triangles whose vertices are (0, 0), (0, 5), (11, 9) and (0, 0), (7, 0), (11, 9). If we take the side on the axis of y as the base of the first, its base is 5 and its altitude is 11; hence its area $= 27\frac{1}{2}$. Taking the side on the axis of x as the base of the second triangle, its base is 7 and its altitude is 9; hence its area is $31\frac{1}{2}$. Therefore the area of the quadrilateral is $27\frac{1}{2} + 31\frac{1}{2}$, or 59.

51. What point in the line $5x - 4y - 28 = 0$ is equidistant from the points (1, 5) and (7, −3)?

If (x, y) is equidistant from (1, 5) and (7, −3), we have

$$(x-1)^2 + (y-5)^2 = (x-7)^2 + (y+3)^2, \text{ or } 3x - 4y = 8. \quad (1)$$

The intersection of (1) and $5x - 4y - 28 = 0$ is $(10, 5\frac{1}{2})$, which is therefore the required point.

52. Prove that the diagonals of a square are perpendicular to each other.

Take two sides of the square as axes of reference, and denote the length of a side by a; then the vertices are $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$. One diagonal passes through $(0, 0)$ and (a, a), and therefore its equation is $y = x$; the intercepts of the other are each a, and therefore its equation is $\frac{x}{a} + \frac{y}{a} = 1$ or $y = -x + a$. The slopes of these diagonals being negative reciprocals, the diagonals are at right angles.

53. Prove that the line joining the middle points of two sides of a triangle is parallel to the third side.

Take the third side as the axis of x, and denote the altitude of the triangle by c; then the ordinate of the middle point of each of the two sides will be $\frac{1}{2}c$, and the line through them will be $y = \frac{1}{2}c$, which is parallel to the axes of x, on which the third side lies.

54. What is the geometric meaning of the equation $xy = 0$?

The equation $xy = 0$ is satisfied when $x = 0$ and when $y = 0$; $x = 0$ represents the axis of y, and $y = 0$ represents the axis of x; hence, $xy = 0$ represents the co-ordinate axes.

55. Show that the three points $(3a, 0)$, $(0, 3b)$, $(a, 2b)$ are in a straight line.

The equation of the line through $(3a, 0)$ and $(0, 3b)$ is

$$\frac{x}{3a} + \frac{y}{3b} = 1, \qquad (1)$$

which is satisfied for $x = a$ and $y = 2b$; hence, $(a, 2b)$ is on line (1).

56. Show that the three lines $5x + 3y - 7 = 0$, $3x - 4y - 10 = 0$, and $x + 2y = 0$ meet in a point.

The point of intersection of $5x + 3y - 7 = 0$ and $3x - 4y - 10 = 0$ is $(2, -1)$, whose co-ordinates satisfy $x + 2y = 0$; hence the three lines meet in the point $(2, -1)$.

57. What must be the value of a in order that the three lines $3x + y - 2 = 0$, $2x - y - 3 = 0$, and $ax + 2y - 3 = 0$ may meet in a point?

The point of intersection of $3x + y - 2 = 0$ and $2x - y - 3 = 0$ is $(1, -1)$. For $x = 1$, and $y = -1$ in $ax + 2y - 3 = 0$ we have $a = 5$; hence, if $a = 5$, the three lines meet in the point $(1, -1)$.

58. What straight lines are represented by the equation

$$x^2 + (a - b)x - ab = 0?$$

The equation may be written in the form

$$(x + a)(x - b) = 0,$$

the locus of which is the two straight lines $x + a = 0$ and $x - b = 0$ which are both parallel to the axes of y.

59. What straight lines are represented by the equation

$$xy + bx + ay + ab = 0?$$

The equation may be written in the form

$$(x+a)(y+b)=0,$$

of which the locus is the two straight lines $x+a=0$ and $y+b=0$. The first is parallel to the axis of y, and the second to the axis of x.

60. What straight lines are represented by the equation

$$x^2y=xy^2?$$

The equation may be written in the form

$$xy(x-y)=0,$$

of which the locus is the three straight lines $x=0$, $y=0$, and $y=x$. The first is the axis of y, the second the axis of x, and the third bisects the first and third quadrants.

61. What straight lines are represented by the equation

$$14x^2-5xy-y^2=0?$$

The equation may be written in the form

$$(2x-y)(7x+y)=0,$$

of which the locus is the two straight lines $2x-y=0$ and $7x+y=0$.

62. Prove that the locus of the vertex of a triangle having the base and the area constant is a straight line, and obtain its equation.

Take the base of the triangle as the axis of x, and the left-hand adjacent vertex as the origin. Since the base and area are both constant, the altitude must be constant. Denote this altitude by h; then in every position of the vertex its ordinate is h, and the equation of its locus is $y=h$. Since $y=h$ is of the first degree, its locus is a straight line.

63. Prove that the locus of a point equidistant from the points (x_1, y_1) and (x_2, y_2) is a straight line, and obtain its equation.

If (x, y) is equidistant from (x_1, y_1) and (x_2, y_2), then, by [1], we have

$$(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2,$$

or

$$2(x_2-x_1)x+2(y_2-y_1)y=x_2^2-x_1^2+y_2^2-y_1^2. \qquad (1)$$

Since (1) is of the first degree, its locus is a straight line.

64. Prove that the locus of a point at the distance d from the line $Ax + By + C = 0$ consists of two straight lines, and obtain their equations.

Reducing $Ax + By + C = 0$ to the normal form, we have

$$\mp \frac{Ax + By}{\sqrt{A^2 + B^2}} = \mp \frac{-C}{\sqrt{A^2 + B^2}}, \tag{1}$$

in which the upper or lower sign is to be taken according as C is positive or negative.

The second member of (1) is the distance of its locus from the origin. If we add d to, and subtract it from, this member, we have the equations of lines parallel to (1), and at the distance d from it. Hence the required equations are

$$\mp \frac{Ax + By}{\sqrt{A^2 + B^2}} = \mp \frac{-C}{\sqrt{A^2 + B^2}} \pm d,$$

or $$Ax + By + C \pm d\sqrt{A^2 + B^2} = 0.$$

65. Prove that the locus of a point so moving that the sum of its distances from the axes shall be constant and equal to k is a straight line, and obtain its equation.

If we consider the direction, or sign, of these distances as well as their length, they will be denoted by x and y respectively; hence we have $x + y = k$ as the required equation, of which the locus is a straight line.

66. Prove that the locus of a point so moving that the sum of its distances from the lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ shall be constant and equal to k, is a straight line, and obtain its equation.

If we regard C and C' as each positive, by [12] the distance of (x, y) from $Ax + By + C = 0$ is $\frac{Ax + By + C}{\sqrt{A^2 + B^2}}$, and its distance from $A'x + B'y + C' = 0$ is $\frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}$.

Hence the equation of the locus is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} + \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = k. \tag{1}$$

Since (1) is of the first degree, its locus is a straight line.

67. Prove that the locus of the vertex of a triangle, having given the base and the difference of the squares of the other sides, is a straight line, and obtain its equation.

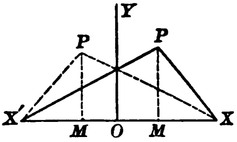

FIG. 40.

Let $XX'P$, be the triangle. Take the middle point O of the base as the origin, XX' as the axis of x, and OY as the axis of y. Denote P by (x, y), the base by b, and the constant difference by k^2. Then,

$$X'P^2 - PX^2 = \pm k^2, \qquad (1)$$

But $X'P^2 = X'M^2 + MP^2 = (X'O \pm OM)^2 + MP^2 = (\frac{1}{2}b + x)^2 + y^2$.

$$PX^2 = MX^2 + MP^2 = (\tfrac{1}{2}b - x)^2 + y^2.$$

Substituting in (1) we have

$$(\tfrac{1}{2}b + x)^2 + y^2 - [(\tfrac{1}{2}b - x)^2 + y^2] = \pm k^2, \text{ or } 2bx = \pm k^2,$$

of which the locus is two straight lines parallel to the axis of y.

Ex. 18. Page 64.

1. Find the equation of a line passing through the origin, and the intersection of the lines $3x - 2y + 17 = 0$, $x + 4y - 27 = 0$.

By § 51 the required equation may be written in the form

$$3x - 2y + 17 + k(x + 4y - 27) = 0. \qquad (1)$$

Since (1) passes through (0, 0), we must have

$$17 - 27k = 0, \text{ or } k = 17 \div 27.$$

Substituting this value of k in (1), we obtain after reduction

$$7x + y = 0.$$

as the required equation.

2. Find the equation of a line passing through the intersection of the lines $3x - 2y + 17 = 0$, $x + 4y - 27 = 0$, and parallel to the line $x + 2y + 3 = 0$.

The intersection of the lines $3x - 2y + 17 = 0$ and $x + 4y - 27 = 0$ is the point $(-1, 7)$.

By § 46, the equation of the line through $(-1, 7)$ parallel to $y = -\frac{1}{2}x - \frac{3}{2}$ is

$$y - 7 = -\tfrac{1}{2}(x + 1), \text{ or } y = -\tfrac{1}{2}x + 6\tfrac{1}{2}.$$

3. Find the equation of a line passing through the intersection of the lines, $3x - 2y + 17 = 0$, $x + 4y - 27 = 0$, and perpendicular to the line $6x - 5y = 0$.

By § 46, the equation of the line through $(-1, 7)$ perpendicular to $y = \frac{6}{5}x$ is

$$y - 7 = -\tfrac{5}{6}(x + 1), \text{ or } y = -\tfrac{5}{6}x + 6\tfrac{1}{6}.$$

4. Find the equation of a line passing through the intersection of the lines $3x - 2y + 17 = 0$, $x + 4y - 27 = 0$, and equally inclined to the two axes.

If the line is equally inclined to the two axes, its slope must be $+1$ or -1.

By [5], the equations of the lines through $(-1, 7)$ having the slopes ± 1, are $y - 7 = \pm 1(x + 1)$, or $y = x + 8$ and $y = -x + 6$.

5. Find the equation of a line parallel to the line $x = y$, and passing through the intersection of the lines $y = 2x + 1$ and $y + 3x = 11$.

The intersection of $y = 2x + 1$ and $y + 3x = 11$ is $(2, 5)$.

By § 46, the equation of the line through $(2, 5)$ parallel to $y = x$ is

$$y - 5 = 1(x - 2), \text{ or } y = x + 3.$$

6. Find the equation of a straight line joining $(2, 3)$ to the intersection of the lines $2x + 3y + 1 = 0$ and $3x - 4y = 5$.

By § 51, the required equation may be written in the form

$$2x + 3y + 1 + k(3x - 4y - 5) = 0. \qquad (1)$$

Since (1) passes through $(2, 3)$, we must have

$$4 + 9 + 1 + k(6 - 12 - 5) = 0, \text{ or } k = \tfrac{14}{11}.$$

Substituting this value of k in (1), we obtain, after reduction,

$$64x - 23y - 59 = 0.$$

7. Find the equation of a straight line joining $(0, 0)$ to the intersection of the lines $5x - 2y + 3 = 0$ and $13x + y = 1$.

By § 51, the required equation may be written in the form

$$5x - 2y + 3 + k(13x + y - 1) = 0. \qquad (1)$$

Since (1) passes through $(0, 0)$, we must have

$$3 - k = 0, \text{ or } k = 3.$$

Substituting this value of k in (1), we obtain as the required equation

$$44x + y = 0.$$

8. Find the equation of a straight line joining (1, 11) to the intersection of the lines $2x+5y-8=0$ and $3x-4y=8$.

By § 51, the required equation may be written in the form

$$2x+5y-8+k(3x-4y-8)=0, \qquad (1)$$

Since (1) passes through (1, 11), we must have

$$2+55-8+k(3-44-8)=0, \text{ or } k=1.$$

Substituting this value of k in (1), we obtain

$$5x+y-16=0.$$

9. Find the equation of the straight line passing through the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, and also passing through the origin.

By § 51, the required equation may be written in the form

$$Ax+By+C+k(A'x+B'y+C')=0. \qquad (1)$$

Since (1) passes through (0, 0), we must have

$$C+kC'=0, \text{ or } k=-(C\div C').$$

Substituting this value of k in (1), we obtain

$$(AC'-A'C)x+(BC'-B'C)y=0.$$

10. Find the equation of the straight line passing through the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, and also drawn parallel to the axis of x.

By § 51, the required equation may be written in the form

$$Ax+By+C+k(A'x+B'y+C')=0,$$

or

$$(A+kA')x+(B+kB')y+C+kC'=0. \qquad (1)$$

The slope of (1) is $-\frac{A+kA'}{B+kB'}$, which must equal zero as the line is parallel to the axis of x.

Hence $A+kA'=0$, or $k=-(A\div A')$.

Substituting this value of k in (1) we obtain

$$(A'B-AB')y+A'C-AC'=0,$$

as the required equation.

11. Find the equation of the straight line passing through the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, and also passing through the point (x_1, y_1).

By § 51, the required equation may be written in the form

$$Ax + By + C = -k(A'x + B'y + C'). \qquad (1)$$

Since (1) passes through (x_1, y_1), we must have

$$Ax_1 + By_1 + C = -k(A'x_1 + B'y_1 + C'). \qquad (2)$$

Dividing (1) by (2) we obtain as the required equation

$$\frac{Ax + By + C}{Ax_1 + By_1 + C} = \frac{A'x + B'y + C'}{A'x_1 + B'y_1 + C'}.$$

12. Find the equation of a straight line passing through the intersection of $5x - 4y + 3 = 0$ and $7x + 11y - 1 = 0$, and cutting on the axis of y an intercept equal to 6.

By § 51, the required equation may be written in the form

$$5x - 4y + 3 = -k(7x + 11y - 1). \qquad (1)$$

Since $b = 6$, (1) passes through (0, 6), and we must have

$$-24 + 3 = -k(66 - 1). \qquad (2)$$

Dividing (1) by (2), and reducing the result we obtain

$$472x - 29y + 174 = 0.$$

13. Find the equation of a straight line passing through the intersection of $y = 7x - 4$ and $y = -2x + 5$, and forming with the axis of x the angle 60°.

The intersection of $y = 7x - 4$ and $y = -2x + 5$ is (1, 3).

Here $m = \tan\gamma = \tan 60° = \sqrt{3}$.

By § 46, the equation of a line through (1, 3) having a slope $\sqrt{3}$, is

$$y - 3 = \sqrt{3}(x - 1), \text{ or } y = \sqrt{3}x + 3 - \sqrt{3}.$$

14. The distance of a straight line from the origin is 5; and it passes through the intersection of the lines

$$3x - 2y + 11 = 0 \text{ and } 6x + 7y - 55 = 0.$$

What is its equation?

By § 51, the required equation may be written in the form

$$3x - 2y + 11 + k(6x + 7y - 55) = 0. \qquad (1)$$

Reducing (1) to the normal form, we have

$$\frac{(3 + 6k)x}{\sqrt{85k^2 + 8k + 13}} + \frac{(7k - 2)y}{\sqrt{85k^2 + 8k + 13}} = \frac{55k - 11}{\sqrt{85k^2 + 8k + 13}}. \qquad (2)$$

Since $p = 5$, we put the second member of (2) equal to 5, and obtain, after squaring and clearing of fractions,

$$900\,k^2 - 1410\,k = 204. \tag{3}$$

Solving (3) for k, we obtain

$$k = \tfrac{17}{10}, \text{ and } -\tfrac{2}{15}.$$

Substituting these values successively in (1), we obtain

$$4x + 3y - 25 = 0 \text{ and } 3x - 4y + 25 = 0,$$

as the required equations.

15. What is the equation of the straight line passing through the intersection of $bx + ay = ab$ and $y = mx$, and perpendicular to the former line?

By § 51, the required equation may be written in the form

$$bx + ay - ab + k(y - mx) = 0. \tag{1}$$

Reducing (1) to the form $y = mx + b$, we have

$$y = \frac{km - b}{a + k}x + \frac{ab}{a + k}. \tag{2}$$

The slope of $bx + ay = ab$ is $-(b \div a)$. Since this line is perpendicular to (2), we must have

$$\frac{a}{b} = \frac{km - b}{a + k}, \text{ or } k = \frac{a^2 + b^2}{bm - a}.$$

Substituting this value of k in (1), we obtain

$$bx + ay - ab + \frac{a^2 + b^2}{bm - a}(y - mx) = 0,$$

which may be reduced to the form

$$\frac{y}{a} - \frac{x}{b} = \frac{bm - a}{b + am}.$$

16. Prove that the lines $y = 2x + 1$, $y = x + 3$, $y = -5x + 15$ are concurrent (or pass through one point).

The intersection of $y = 2x + 1$ and $y = x + 3$ is (2, 5), and the co-ordinates of this point satisfy $y = -5x + 15$; hence the three lines are concurrent.

17. Prove that the lines

$$4x - 2y - 3 = 0,\ 3x - y + \tfrac{1}{2} = 0,\ 5x - 2y - 1 = 0$$

are concurrent (or pass through one point).

The intersection of $4x - 2y - 3 = 0$ and $3x - y + \frac{1}{2} = 0$ is $(-2, -\frac{11}{2})$, and the co-ordinates of this point satisfy $5x - 2y - 1 = 0$; hence the three lines are concurrent.

18. Prove that the lines

$$2x - y = 5,\ 3x - y = 6,\ 4x - y = 7$$

are concurrent (or pass through one point).

The intersection of $2x - y = 5$ and $3x - y = 6$ is $(1, -3)$, and the co-ordinates of this point satisfy $4x - y = 7$; hence the three lines are concurrent.

19. What is the value of m if the lines

$$\frac{x}{a} + \frac{y}{b} = 1,\ \frac{x}{b} + \frac{y}{a} = 1,\ y = mx$$

meet in one point?

The intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ is $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

Substituting the co-ordinates of this point in $y = mx$, we obtain $m = 1$.

20. When do the straight lines $y = mx + b$, $y = m'x + b'$, $y = m''x + b''$ pass through one point?

The intersection of $y = mx + b$ and $y = m'x + b'$ is

$$\left(\frac{b' - b}{m - m'}, \frac{mb' - bm'}{m - m'}\right).$$

If $y = m''x + b''$ pass through this point, we must have

$$\frac{mb' - bm'}{m - m'} = \frac{m''b' - m''b}{m - m'} + b'',$$

which by clearing of fractions, and adding $b''m''$ to each member, may be reduced to the form

$$\frac{m'' - m}{m'' - m'} = \frac{b'' - b}{b'' - b'}. \qquad (1)$$

Hence these three lines pass through one point when equation (1) is satisfied.

21. Prove that the three altitudes of a triangle meet in one point.

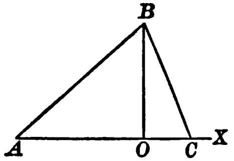

FIG. 41.

Let ABC be any triangle. Draw BO perpendicular to AC, and take AX and OB as axes of reference. Let $OC = a$, $OB = b$, and $OA = c$; then the point A is $(-c, 0)$, C is $(a, 0)$, and B is $(0, b)$.

The equation of AB, or the line whose intercepts are $-c$ and b, is, by [7],

$$\frac{x}{-c} + \frac{y}{b} = 1, \text{ or } y = \frac{b}{c}x + b. \tag{1}$$

The equation of the perpendicular to (1) from C, or $(a, 0)$, is, by § 46,

$$y = -\frac{c}{b}(x - a), \text{ or } y = -\frac{c}{b}x + \frac{ac}{b}. \tag{2}$$

The equation of BC, or the line whose intercepts are a and b, is, by [7],

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ or } y = -\frac{b}{a}x + b. \tag{3}$$

The equation of the perpendicular to (3) from A, or $(-c, 0)$, is, by § 46,

$$y = \frac{a}{b}(x + c), \text{ or } y = \frac{a}{b}x + \frac{ac}{b}. \tag{4}$$

Since (2) and (4) have the same intercepts on the axis of y, the altitudes (2) and (4) cut the altitude $x = 0$ at the same point. The common point of the three altitudes is $\left(0, \frac{ac}{b}\right)$.

22. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

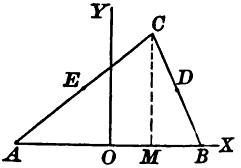

FIG. 42.

Let ABC be any triangle. Draw OY perpendicular to AB at its middle point, and take OX and OY as axes of reference.

Let $OA = OB = a$, $OM = c$, $MC = d$; then A is $(-a, 0)$, B is $(a, 0)$, and C is (c, d). Hence the middle point D, by [2], is $\left(\frac{a+c}{2}, \frac{d}{2}\right)$, and the middle point E is $\left(\frac{c-a}{2}, \frac{d}{2}\right)$.

By [4] the equation of BC, or the line through $(a, 0)$ and (c, d), is

$$\frac{y}{x-a} = \frac{d}{c-a}, \text{ or } y = \frac{d}{c-a}x - \frac{ad}{c-a}. \tag{1}$$

By § 46, the line perpendicular to (1) and passing through D, $\left(\frac{a+c}{2}, \frac{d}{2}\right)$, is

$$y - \frac{d}{2} = \frac{a-c}{d}\left(x - \frac{a+c}{2}\right), \text{ or } y = \frac{a-c}{d}x + \frac{d^2 - a^2 + c^2}{2d}. \tag{2}$$

By [4], the equation of AC, or the line through $(-a, 0)$ and (c, d), is

$$\frac{y}{x+a} = \frac{d}{c+a}, \text{ or } y = \frac{d}{c+a}x + \frac{da}{c+a}. \tag{3}$$

By § 46, the line perpendicular to (3) and passing through E, $\left(\frac{c-a}{2}, \frac{d}{2}\right)$, is

$$y - \frac{d}{2} = -\frac{c+a}{d}\left(x - \frac{c-a}{2}\right), \text{ or } y = -\frac{c+a}{d}x + \frac{d^2 - a^2 + c^2}{2d}. \tag{4}$$

Since the perpendiculars (2) and (4) have the same intercepts on the axis of y, the three perpendiculars (2), (4), and $x = 0$, meet in a point.

23. Prove that the three medians of a triangle meet in one point. Show also that this point is one of the two points of trisection for each median.

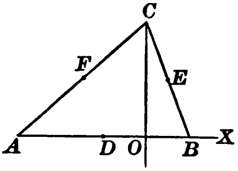

FIG. 43.

Let ABC be any triangle. Draw CO perpendicular to AB, and take AOX and OC as axes. Let $OB = a$, $OA = c$, $OC = b$; then A is $(-c, 0)$, B is $(a, 0)$, and C is $(0, b)$.

By [2] the mid-point D is $\left(\frac{a-c}{2}, 0\right)$, the mid-point E is $(\frac{1}{2}a, \frac{1}{2}b)$, and the mid-point F is $(-\frac{1}{2}c, \frac{1}{2}b)$.

By [4] the equation of the median from B, $(a, 0)$, to F, $(-\frac{1}{2}c, \frac{1}{2}b)$, is

$$\frac{y}{x-a} = \frac{\frac{1}{2}b}{-\frac{1}{2}c - a}, \text{ or } (c + 2a)y = -bx + ba. \tag{1}$$

The equation of the median from A, $(-c, 0)$, to E, $(\frac{1}{2}a, \frac{1}{2}b)$, is

$$\frac{y}{x+c} = \frac{\frac{1}{2}b}{\frac{1}{2}a + c}, \text{ or } (a + 2c)y = bx + bc. \tag{2}$$

The equation of the median from C, $(0, b)$, to D, $\left(\frac{a-c}{2}, 0\right)$, is

$$\frac{y-b}{x} = \frac{-b}{\frac{1}{2}(a-c)}, \text{ or } (a-c)y = -2bx + (a-c)b. \tag{3}$$

The intersection of the medians (1) and (2) is $\left(\frac{a-c}{3}, \frac{b}{3}\right)$, and the co-ordinates of this point satisfy the median (3); hence the three medians meet in a point.

If (x, y) be the point that divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $m:n$, we have

$$x = \frac{mx_2 + nx_1}{m+p}, \quad y = \frac{my_2 + ny_1}{m+n}. \qquad [3]$$

Let (x_1, y_1) be E, $(\frac{1}{2}a, \frac{1}{2}b)$, and (x_2, y_2) be A, $(-c, 0)$, and $m:n = 1:2$; then $x = \frac{a-c}{3}$, $y = \frac{b}{3}$, or the point of trisection of the median AE nearest E is the point of intersection of the medians.

Let (x_1, y_1) be D, $\left(\frac{a-c}{2}, 0\right)$, and (x_2, y_2) be C, $(0, b)$, and $m:n = 1:2$; then $x = \frac{a-c}{3}$, $y = \frac{b}{3}$ as before.

So also if (x_1, y_1) be F, $(-\frac{1}{2}c, \frac{1}{2}b)$, and (x_2, y_2) be B, $(a, 0)$, and $m:n = 1:2$, we have $x = \frac{a-c}{3}$, $y = \frac{b}{3}$.

24. Prove that the bisectors of the three angles of a triangle meet in one point.

Let ABC be any triangle. Let XX' and YY', which intersect within the triangle, be taken as axes. Let the normal equations of AB, BC, CA be respectively

$$x\cos a + y\sin a = p, \tag{1}$$
$$x\cos a' + y\sin a' = p', \tag{2}$$
$$x\cos a'' + y\sin a'' = p''. \tag{3}$$

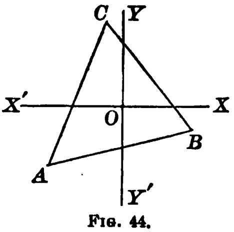

FIG. 44.

By § 54, the equation of the bisector of the angle ABC in which the origin lies is

$$x\cos a + y\sin a - p = x\cos a' + y\sin a' - p. \tag{4}$$

For like reason the bisectors of BCA and CAB are respectively

$$x\cos a' + y\sin a' - p' = x\cos a'' + y\sin a'' - p'', \tag{5}$$

and

$$x\cos a'' + y\sin a'' - p'' = x\cos a + y\sin a - p. \tag{6}$$

Now from either two of the three equations (4), (5), (6) the third may readily be obtained; hence the three bisectors have a common point.

25. The vertices of a triangle are (2, 1), (3, −2), (−4, −1). Find the lengths of its altitudes. Is the origin within or without the triangle?

By [4], the equation of the side through (2, 1) and (3, −2) is

$$\frac{y-1}{x-2} = \frac{-2-1}{3-2} \text{ or } y + 3x = 7. \tag{1}$$

By [12], the distance from (−4, −1) to side (1) is

$$\frac{-1-12-7}{\sqrt{10}} = -2\sqrt{10}.$$

By [4], the equation of the side through (3, −2) and (−4, −1) is

$$\frac{y+2}{x-3} = \frac{-1+2}{-4-3}, \text{ or } 7y + x = -11. \tag{2}$$

By [12], the distance from (2, 1) to side (2) is

$$\frac{-7-2-11}{\sqrt{50}} = -2\sqrt{2}.$$

By [4], the equation of the side through (−4, −1) and (2, 1) is

$$\frac{y+1}{x+4} = \frac{1+1}{2+4}, \text{ or } 3y - x = 1. \tag{3}$$

By [12], the distance from (3, −2) to side (3) is

$$\frac{-6-3-1}{\sqrt{10}} = -\sqrt{10}.$$

Hence its altitudes are $2\sqrt{10}$, $2\sqrt{2}$, and $\sqrt{10}$. Since the formula gives a negative result for each altitude, each vertex and the origin lie on the same side of the opposite side of the triangle; hence the origin must be within the triangle.

26. The equations of the sides of a triangle are

$$3x + y + 4 = 0,\ 3x - 5y + 34 = 0,\ 3x - 2y + 1 = 0.$$

Find the lengths of its altitudes.

The intersection of $3x + y + 4 = 0$ and $3x - 5y + 34 = 0$ is $(-3, 5)$. By [12], the distance from $(-3, 5)$ to $3x - 2y + 1 = 0$ is

$$\frac{9 + 10 - 1}{\sqrt{13}} = \frac{18}{\sqrt{13}} = \tfrac{18}{13}\sqrt{13}.$$

The intersection of $3x - 5y + 34 = 0$ and $3x - 2y + 1 = 0$ is $(7, 11)$. By [12], the distance from $(7, 11)$ to $3x + y + 4 = 0$ is

$$\frac{-21 - 11 - 4}{\sqrt{10}} = -\tfrac{18}{5}\sqrt{10}.$$

The intersection of $3x - 2y + 1 = 0$ and $3x + y + 4 = 0$ is $(-1, -1)$. By [12], the distance from $(-1, -1)$ to $3x - 5y + 34 = 0$ is

$$\frac{3 - 5 - 34}{\sqrt{34}} = -\tfrac{18}{17}\sqrt{34}.$$

Hence the altitudes are $\tfrac{18}{13}\sqrt{13}$, $\tfrac{18}{5}\sqrt{10}$, and $\tfrac{18}{17}\sqrt{34}$. The origin is without the triangle since one vertex and the origin are on opposite sides of a side.

27. What are the equations of the lines bisecting the angles between the lines $3x - 4y + 7 = 0$ and $4x - 3y + 17 = 0$?

Putting these equations in the form of $Ax + By = C$, we have

$$-3x + 4y = 7 \text{ and } -4x + 3y = 17.$$

Hence by [14] the equations of the bisectors are

$$\frac{-3x + 4y - 7}{5} = \pm\frac{-4x + 3y - 17}{5},$$

or $$x + y + 10 = 0 \text{ and } 7x - 7y + 24 = 0,$$

of which the first bisects the angle in which the origin lies.

28. What are the equations of the line bisecting the angles between the lines $3x + 4y - 9 = 0$ and $12x + 5y - 3 = 0$?

By [14] the equations of the bisectors are

$$\frac{3x + 4y - 9}{5} = \pm\frac{12x + 5y - 3}{13},$$

or $$7x - 9y + 34 = 0 \text{ and } 9x + 7y - 12 = 0.$$

29. What are the equations of the line bisecting the angles between the lines $y = 2x - 4$ and $2y = x + 10$?

Here the equations in the form $Ax + By = C$ are

$$2x - y = 4 \text{ and } -x + 2y = 10.$$

Hence, by [14], the equations of the bisectors are

$$\frac{2x-y-4}{\sqrt{5}}=\pm\frac{-x+2y-10}{\sqrt{5}},$$

or $$x-y+2=0 \text{ and } x+y-14=0,$$

of which the first bisects the angle in which the origin lies.

30. What are the equations of the lines bisecting the angles between the lines $x+y=2$ and $x-y=0$?

By [14], the equations of the bisectors are

$$\frac{x+y-2}{\sqrt{2}}=\pm\frac{x-y}{\sqrt{2}},$$

or $$y=1 \text{ and } x=1.$$

31. What are the equations of the lines bisecting the angles between the lines $y=mx+b$ and $y=m'x+b'$?

Here the equations in the form of $Ax+By-C=0$ are

$$\pm(y-mx-b)=0 \text{ and } \pm(y-m'x-b')=0,$$

the upper or lower sign being used, according as b and b' are positive or negative.

Hence, by [14], the equations of the bisectors are

$$\frac{\pm(y-mx-b)}{\sqrt{m^2+1}}=\pm\frac{\pm(y-m'x-b')}{\sqrt{m^2+1}},$$

in which, after the proper signs are taken in the numerators, the + sign before the fraction gives the equation of the bisector of the angle in which the origin lies, and the − sign that of the other bisector.

32. Prove that the bisectors of the two supplementary angles formed by two intersecting lines are perpendicular to each other.

The equations of the bisectors of the angles included between any two lines $x\cos a+y\sin a=p$ and

$$x\cos a'+y\sin a'=p'$$

are $$x\cos a+y\sin a'-p=\pm(x\cos a'+y\sin a'-p'). \qquad (1)$$

Now the slopes of lines (1) are respectively

$$-\frac{\cos a-\cos a'}{\sin a-\sin a'} \text{ and } -\frac{\cos a+\cos a'}{\sin a+\sin a'}.$$

Multiplying these slopes together, we have

$$\frac{\cos^2 a - \cos^2 a'}{\sin^2 a - \sin^2 a'} = \frac{(1-\sin^2 a)-(1-\sin^2 a')}{\sin^2 a - \sin^2 a'}$$

$$= \frac{-(\sin^2 a - \sin^2 a')}{\sin^2 a - \sin^2 a'} = -1.$$

Hence the slopes of these bisectors satisfy the condition $mm' = -1$, and the bisectors are perpendicular.

Ex. 19. Page 68.

1. Describe the position of the two straight lines represented by the equation $Ax^2 + Cxy + By^2 + Dx + Ey + F = 0$, where

(i.) $A = C = D = 0$, (ii.) $B = C = E = 0$.

(i.) If $A = C = D = 0$, the equation becomes

$$By^2 + Ey + F = 0. \qquad (1)$$

Solving (1) for y, we obtain

$$y = \frac{-E \pm \sqrt{E^2 - 4BF}}{2B}. \qquad (2)$$

If $E^2 - 4BF > 0$, the locus of (2) is two lines parallel to the axis of x. If $E^2 - 4BF = 0$, these two lines are coincident. If

$$E^2 - 4BF < 0,$$

(2) has no real locus.

2. When will the equation $axy + bx + cy + d = 0$ represent two straight lines?

Comparing this equation with the general equation (1) of § 57, we have $A = 0$, $B = 0$, $C = a$, $D = b$, $E = c$, $F = d$.

Substituting these values in (3) of § 57, we obtain

$$da^2 - abc = 0, \text{ or } da = bc.$$

Hence the given equation will represent straight lines when $da = bc$.

3. Find the conditions that the straight lines represented by the equation $Ax^2 + Cxy + By^2 = 0$ may be real; imaginary; coincident; perpendicular to each other.

Solving the equation for y, we obtain

$$y = \frac{-2Ax}{C \pm \sqrt{C^2 - 4AB}}. \qquad (1)$$

Now (1) represents two distinct lines through the origin when $C^2 - 4AB > 0$, two coincident lines when $C^2 - 4AB = 0$, and no real locus when $C^2 - 4AB < 0$.

From § 56, we have, ϕ denoting the angle between the lines,

$$\tan\phi = \frac{\sqrt{C^2 - 4AB}}{A + B}. \qquad (2)$$

From (2) we see that if $A + B = 0$, $\tan\phi = \infty$, or $\phi = 90°$; that is, the lines are perpendicular to each other if $A + B = 0$, and conversely.

4. Show that the two straight lines $x^2 - 2xy\sec\theta + y^2 = 0$ make the angle θ with each other.

Here A of § 56 is 1, $C = -2\sec\theta$, $B = 1$.

Substituting these values in the value of $\tan\phi$ (§ 56), we have

$$\tan\phi = \frac{\sqrt{C^2 - 4AB}}{A + B} = \frac{\sqrt{4\sec^2\theta - 4}}{2} = \tan\theta.$$

Hence $\quad \phi = \theta.$

5. Show that $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0$ represents straight lines, and find their separate equations.

Comparing this equation with the general equation, (1) of § 57, we obtain $A = 1$, $B = -3$, $C = -2$, $D = 2$, $E = -2$, $F = 1$. By trial we find that these values satisfy the condition (3) of § 57; hence the equation represents two loci of the first order. Solving the equation for x, we obtain

$$x = y - 1 \pm 2y,$$

or $\quad x - 3y + 1 = 0$ and $x + y + 1 = 0$,

as the required equations.

6. Show that $x^2 - 4xy + 5y^2 - 6y + 9 = 0$ represents two loci of the first order, and find their separate equations.

Here $A = 1$, $B = 5$, $C = -4$, $D = 0$, $E = -6$, $F = 9$.

These values satisfy condition (3) of § 57; hence the equation represents two loci of the first order. Solving for x, we obtain

$$x = 2y \pm (y - 3)\sqrt{-1}.$$

Since each of these simple equations contains an imaginary term, neither has a real locus.

7. Show that $x^2 - 4xy + 3y^2 + 6y - 9 = 0$ represents straight lines, and find their separate equations.

Here $A = 1$, $B = 3$, $C = -4$, $D = 0$, $E = 6$, $F = -9$.

These values satisfy condition (3) of § 57; hence the equation represents two loci of the first order. Solving for x, we obtain

$$x = 3y - 3, \text{ and } x = y + 3,$$

as the separate equations of these loci.

8. Show that the equation $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$ represents two straight lines, and find the angle between them.

Here $A = 1$, $B = -6$, $C = 1$, $D = 7$, $E = 31$, $F = -18$.

These values satisfy condition (3) of § 57; hence the given equation represents two loci of the first order. Solving for x, we obtain

$$x = 2y - 9, \text{ and } x = -3y + 2,$$

or

$$y = \tfrac{1}{2}x + \tfrac{9}{2}, \text{ and } y = -\tfrac{1}{3}x + \tfrac{2}{3}. \qquad (1)$$

The slopes of lines (1) are $\frac{1}{2}$ and $-\frac{1}{3}$; hence

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1, \text{ whence } \phi = 45^\circ.$$

9. Determine the value of K for which

$$12x^2 - 10xy + 2y^2 + 11x - 5y + K = 0 \qquad (1)$$

will represent a pair of straight lines. Are the lines real or imaginary?

Here $A = 12$, $B = 2$, $C = -10$, $D = 11$, $E = -5$, $F = K$.

Substituting these values in (3) of § 57, we obtain

$$K - 2 = 0, \text{ or } K = 2.$$

Hence when $K = 2$, the given equation represents two loci of the first order.

Putting $K = 2$ in (1), and solving for y, we obtain

$$y = 3x + 2, \text{ and } y = 2x + \tfrac{1}{2}. \qquad (2)$$

The loci of (2) are real and distinct lines.

10. Determine the values of K for which

$$12x^2 + Kxy + 2y^2 + 11x - 5y + 2 = 0$$

will represent a pair of straight lines. Are the lines real or imaginary?

Here $A = 12$, $B = 2$, $C = K$, $D = 11$, $E = -5$, $F = 2$.

Substituting these values in (3) of § 57, and solving for K, we obtain

$$K = -10 \text{ and } -\tfrac{35}{2}.$$

Putting $K = -10$, and solving for y, we obtain

$$y = 3x + 2,\ y = 2x + \tfrac{1}{2},$$

whose loci are two real straight lines.

Putting $K = -\frac{35}{2}$, and solving for y, we obtain

$$y = 8x + 2, \text{ and } y = \tfrac{3}{4}x + \tfrac{1}{2},$$

whose loci are two real straight lines.

11. Determine the values of K for which

$$12x^2 + 36xy + Ky^2 + 6x + 6y + 3 = 0$$

will represent a pair of straight lines. Are the lines real or imaginary?

Here $A = 12,\ B = K,\ C = 36,\ D = 6,\ E = 6,\ F = 3.$

Substituting these values in (3) of § 57, we obtain

$$K - 28 = 0, \text{ or } K = 28.$$

Putting $K = 28$, and solving the equation, we obtain imaginary values for x; hence the equation represents no real locus.

12. For what value of K does the equation $Kxy + 5x + 3y + 2 = 0$ represent two straight lines?

Here $A = 0,\ B = 0,\ C = K,\ D = 5,\ E = 3,\ F = 2.$

Substituting these values in (3) of § 57, we obtain

$$2K^2 - 15K = 0, \text{ or } K = 0 \text{ and } \tfrac{15}{2}.$$

For $K = 0$, the equation represents a single straight line. Putting $K = \frac{15}{2}$, and multiplying by 2, we obtain

$$15xy + 10x + 6y + 4 = 0,$$

or $$(5x + 2)(3y + 2) = 0,$$

the locus of which is the lines $5x + 2 = 0$ and $3y + 2 = 0$.

Ex. 20. Page 70.

1. Through a fixed point O any straight line is drawn, meeting two given parallel lines in P and Q; through P and Q straight lines are drawn in fixed directions, meeting in R. Prove that the locus of R is a straight line, and find its equation.

Let O be the fixed point, MP and NQ the parallels, and OPQ any position of the line cutting them.

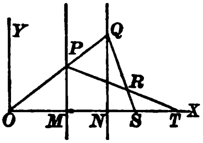

FIG. 45.

Take O as the origin, and OY parallel to MP as the axis of y. Let $OM = a$, $ON = b$, $\tan MOP = m$, slope of $PT = m'$, and slope of $QS = m''$.

Then P is (a, am), and the equation of PT is

$$y - am = m'(x - a). \qquad (1)$$

Also Q is (b, bm), and the equation of QS is

$$y - bm = m''(x - b). \qquad (2)$$

Combining (1) and (2), so as to eliminate m, we obtain

$$(b - a)\,y = m'b\,(x - a) - m''a\,(x - b), \qquad (3)$$

as the equation of the locus of R.

For (3) will hold true for any position of OPQ.

Since (3) is of the first degree, the locus is a straight line.

2. The hypotenuse of a right triangle slides between the axes of x and y, its ends always touching the axes. Find the locus of the vertex of the right angle.

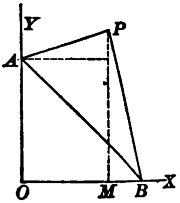

FIG. 46.

Let ABP be the right triangle in any position. Let $BP = a$, $AP = b$, $OB = m$, $OA = n$, $OM = x$, $MP = y$; then B is $(m, 0)$, A is $(0, n)$, and P is (x, y).

By [4] the slopes of PA and PB are respectively $\frac{y - n}{x}$ and $\frac{y}{x - m}$.

Since these lines are perpendicular to each other, we have

$$\frac{y - n}{x} \times \frac{y}{x - m} = -1, \text{ or } \frac{y^2}{x^2} = \frac{(m - x)^2}{(y - n)^2}.$$

Whence, by composition, we have

$$y^2 + (m - x)^2 : x^2 + (y - n)^2 :: y^2 : x^2. \qquad (1)$$

But $y^2 + (m - x)^2 = PB^2 = a^2$, $x^2 + (y - n)^2 = AP^2 = b^2$.
Substituting in (1), we have

$$\frac{y^2}{x^2} = \frac{a^2}{b^2}, \text{ or } y = \pm\frac{a}{b}x.$$

3. Given two fixed points, A and B, one on each of the axes, if U and V are two variable points, one on each axis, so taken that $OU + OV = OA + OB$, find the locus of the intersection of AV and BU.

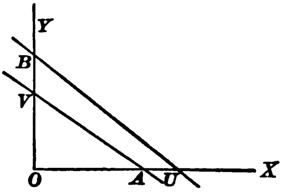

FIG. 47.

Let A and B be the two fixed points and U and V the variable points on the axes OX and OY respectively. Let $OA = a$, $OB = b$.

Since $OU + OV = OA + OB$, we have $OU - OA = OB - OV$, or $AU = VB$.

Let $AU = VB = z$; then $OU = a + z$, and $OV = b - z$. Hence the equations of AV and UB are

$$\frac{x}{a} + \frac{y}{b-z} = 1, \text{ and } \frac{x}{a+z} + \frac{y}{b} = 1.$$

Regarding these equations as simultaneous, and combining them so as to eliminate z, we obtain

$$x + y = a + b, \tag{1}$$

as the equation of the intersection of the two lines.

Since (1) is of the first degree, the locus is a straight line.

4. Find the locus of the middle points of the rectangles which may be inscribed in a given triangle.

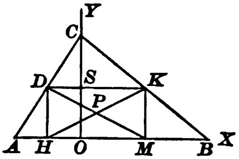

FIG. 48.

Let ABC be any given triangle. Take the base as the axis of x, and the altitude OC as the axis of y. Denote the segments of the base OA and OB by a and b, and the altitude OC by h. Let $MKDH$ be any rectangle inscribed in this triangle; we are required to find the locus of the intersection of its diagonals.

Let $DS = za$; then, from the similarity of the triangles AOC and DSC, $CS = zh$; and from the similarity of CSK and COB, $SK = zb$. Hence, $OS = h - zh$; and M is $(zb, 0)$, K is $(zb, h - zh)$, D is $(-za, h - zh)$, and H is $(-za, 0)$. Hence, by [4], the equations of the lines HK and DM are

$$\frac{y}{x + za} = \frac{h - zh}{zb + za}, \quad \frac{y}{x - zb} = -\frac{h - zh}{az + bz}. \tag{1}$$

From equations (1) we obtain

$$\frac{y}{x+za}=\frac{y}{zb-x}, \text{ or } z=\frac{2x}{b-a}. \tag{2}$$

Substituting this value of z in the last of equations (1), we obtain

$$\frac{2x}{b-a}+\frac{2y}{h}=1.$$

Hence the locus is a straight line through the middle points of the base and altitude.

5. If PP', QQ' are any two parallels to the sides of a given rectangle, find the locus of the intersection of $P'Q$ and PQ'.

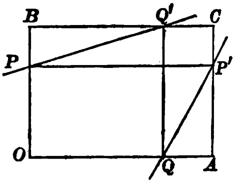

FIG. 49.

Let $OACB$ be the given rectangle, and PP' and QQ' any positions of the parallels to them. Take OA and OB as the axes of x and y respectively.

Let $OA=a$, $OQ=m$, $OB=b$, $OP=n$; then Q is $(m, 0)$, P' is (a, n), P is $(0, n)$, and Q' is (m, b).

By [4] the equations of the line QP' and PQ' are

$$\frac{y}{x-m}=\frac{n}{a-m}, \text{ or } ay-my=nx-mn, \tag{1}$$

and

$$\frac{y-n}{x}=\frac{b-n}{m}, \text{ or } my-mn=bx-nx. \tag{2}$$

Adding (1) and (2), we obtain

$$ay=bx, \text{ or } bx-ay=0 \tag{3}$$

as the equation of the locus.

Hence the locus is the diagonal of the rectangle.

Ex. 21. Page 73.

1. Find the equation of the circle, taking as origin the point B (Fig. 50).

If B is the origin, then C, or (a, b), is $(-r, 0)$.

Substituting these values of a and b in [15], we obtain

$$(x+r)^2+y^2=r^2, \text{ or } x^2+y^2+2rx=0.$$

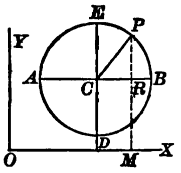

FIG. 50.

2. Find the equation of the circle, taking as origin the point D (Fig. 50).

If D is the origin, then C, or (a, b), is $(0, r)$. Substituting these values of a and b in [15], we obtain

$$x^2 + (y - r)^2 = r^2, \text{ or } x^2 + y^2 = 2ry.$$

3. Find the equation of the circle, taking as origin the point E (Fig. 50).

If E is the origin, then C, or (a, b), is $(0, -r)$. Substituting these values of a and b in [15], we obtain

$$x^2 + (y + r)^2 = r^2, \text{ or } x^2 + y^2 + 2ry = 0.$$

4. Write the equation of the circle, centre $(5, -3)$, radius 10.

Here $r = 10$, and (a, b) is $(5, -3)$. Substituting in [15], we obtain

$$(x - 5)^2 + (y + 3)^2 = 100.$$

5. Write the equation of the circle, centre $(0, -2)$, radius 11.

Here $r = 11$, and (a, b) is $(0, -2)$.
Substituting in [15], we obtain

$$x^2 + (y + 2)^2 = 121.$$

6. Write the equation of the circle, centre $(5, 0)$, radius 5.

Here $r = 5$, and (a, b) is $(5, 0)$.
Substituting in [15], we obtain

$$(x - 5)^2 + y^2 = 25.$$

7. Write the equation of the circle, centre $(-5, 0)$, radius 5.

Here $r = 5$, and (a, b) is $(-5, 0)$.
Substituting in [15], we obtain

$$(x + 5)^2 + y^2 = 25.$$

8. Write the equation of the circle, centre $(2, 3)$, diameter 10.

Here $r = 5$, and (a, b) is $(2, 3)$.
Substituting in [15], we obtain

$$(x - 2)^2 + (y - 3)^2 = 25.$$

9. Write the equation of the circle, centre (h, k), radius $\sqrt{h^2 + k^2}$.

Here $r = \sqrt{h^2 + k^2}$, and (a, b) is (h, k).
Substituting in [15], we obtain

$$(x - h)^2 + (y - k)^2 = h^2 + k^2,$$

or

$$x^2 + y^2 - 2hx - 2ky = 0.$$

10. Determine the centre and radius of the circle

$$x^2 + y^2 - 10x + 12y + 25 = 0.$$

This equation may be written in the form

$$(x^2 - 10x + 25) + (y^2 + 12y + 36) = 36,$$

or $$(x - 5)^2 + (y + 6)^2 = 6^2.$$

Hence, from [15], $r = 6$, and (a, b) is $(5, -6)$.

11. Determine the centre and radius of the circle

$$x^2 + y^2 - 2x - 4y = 0.$$

This equation may be written in the form

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) = 5,$$

or $$(x - 1)^2 + (y - 2)^2 = (\sqrt{5})^2.$$

Hence $r = \sqrt{5}$, and (a, b), the centre, is $(1, 2)$.

12. Determine the centre and radius of the circle

$$3x^2 + 3y^2 - 5x - 7y + 1 = 0.$$

This equation may be written in the form

$$(x^2 - \tfrac{5}{3}x + \tfrac{25}{36}) + (y^2 - \tfrac{7}{3}y + \tfrac{49}{36}) = \tfrac{62}{36},$$

or $$(x - \tfrac{5}{6})^2 + (y - \tfrac{7}{6})^2 = (\tfrac{1}{6}\sqrt{62})^2.$$

Hence $r = \tfrac{1}{6}\sqrt{62}$, and the centre is $(\tfrac{5}{6}, \tfrac{7}{6})$.

13. Determine the centre and radius of the circle

$$x^2 + y^2 - 8x = 0.$$

This equation may be written in the form

$$(x^2 - 8x + 16) + y^2 = 16,$$

or $$(x - 4)^2 + y^2 = 4^2.$$

Hence $r = 4$, and (a, b), the centre, is $(4, 0)$.

14. Determine the centre and radius of the circle $x^2 + y^2 + 8x = 0$.

This equation may be written in the form

$$(x^2 + 8x + 16) + y^2 = 16,$$

or $$(x + 4)^2 + y^2 = 4^2.$$

Hence $r = 4$, and the centre is $(-4, 0)$.

15. Determine the centre and radius of the circle $x^2 + y^2 - 8y = 0$.

This equation may be written in the form

$$x^2 + (y^2 - 8y + 16) = 16,$$

or

$$x + (y - 4)^2 = 4^2.$$

Hence $r = 4$, and (a, b) is $(0, 4)$.

16. Determine the centre and radius of the circle $x^2 + y^2 + 8y = 0$.

This equation may be written in the form

$$x^2 + (y^2 + 8y + 16) = 16,$$

or

$$x^2 + (y + 4)^2 = 4^2.$$

Hence $r = 4$, and (a, b) is $(0, -4)$.

17. Determine the centre and radius of the circle

$$6x^2 - 2y(7 - 3y) = 0.$$

This equation may be written in the form

$$x^2 + (y^2 - \tfrac{7}{3}y + \tfrac{49}{36}) = \tfrac{49}{36},$$

or

$$x^2 + (y - \tfrac{7}{6})^2 = (\tfrac{7}{6})^2.$$

Hence $r = \frac{7}{6}$, and (a, b) is $(0, \frac{7}{6})$.

18. Determine the centre and radius of the circle $x^2 + y^2 = 9k^2$.

Here $r = 3k$, and (a, b) is $(0, 0)$.

19. Determine the centre and radius of the circle

$$(x + y)^2 + (x - y)^2 = 8k^2.$$

Reducing this equation to its simplest form, we obtain

$$x^2 + y^2 = 4k^2.$$

Hence $r = 2k$, and (a, b) is $(0, 0)$.

20. Determine the centre and radius of the circle $x^2 + y^2 = a^2 + b^2$.

Here $r = \sqrt{a^2 + b^2}$, and (a, b) is $(0, 0)$.

21. Determine the centre and radius of the circle

$$x^2 + y^2 = k(x + k).$$

This equation may be written in the form

$$x^2 - kx + \tfrac{1}{4}k^2 + y^2 = \tfrac{5}{4}k^2,$$

or

$$(x - \tfrac{1}{2}k)^2 + y^2 = \tfrac{5}{4}k^2.$$

Hence $r = \frac{1}{2}k\sqrt{5}$, and (a, b) is $(\frac{1}{2}k, 0)$.

22. Determine the centre and radius of the circle $x^2+y^2=hx+ky$.

This equation may be written in the form

$$x^2-hx+\tfrac{1}{4}h^2+y^2-ky+\tfrac{1}{4}k^2=\tfrac{1}{4}(h^2+k^2),$$

or

$$(x-\tfrac{1}{2}h)^2+(y-\tfrac{1}{2}k)^2=\tfrac{1}{4}(h^2+k^2).$$

Hence $r=\frac{1}{2}\sqrt{h^2+k^2}$, and (a, b) is $(\frac{1}{2}h, \frac{1}{2}k)$.

23. When are the circles $x^2+y^2+Dx+Ey+C=0$ and $x^2+y^2+D'x+E'y+C'=0$ concentric?

From § 61, we see that the coefficients of x and y in the equation of a circle determine its centre. Hence these circles are concentric when $D=D'$ and $E=E'$.

24. What is the geometric meaning of the equation

$$(x-a)^2+(y-b)^2=0?$$

This equation is satisfied only when $x=a$ and $y=b$; hence, geometrically, it represents the point (a, b).

25. Find the intercepts of the circles

$$\text{(i.)}\quad x^2+y^2-8x-8y+7=0,$$
$$\text{(ii.)}\quad x^2+y^2-8x-8y+16=0,$$
$$\text{(iii.)}\quad x^2+y^2-8x-8y+20=0.$$

Putting $y=0$ in each case, we have in case (i.) $x^2-8x+7=0$, whence $x=1$ and 7; in case (ii.) $x^2-8x+16=0$, whence $x=4$; in case (iii.) $x^2-8x+20=0$, whence $x=4\pm\sqrt{-4}$.

Putting $x=0$ in each case, we obtain for y values identical with the above values of x.

The geometric meaning of these results is as follows:

Circle (i.) cuts the axis of x in the points $(1, 0)$, $(7, 0)$, and the axis of y in the points $(0, 1)$, $(0, 7)$.

Circle (ii.) touches the axis of x at $(4, 0)$, and the axis of y at $(0, 4)$.

Circle (iii.) does not meet either axis.

This is the meaning of the imaginary values of x and y in case (iii.).

If, however, we wished to make the language of Geometry conform more exactly to that of Algebra, then in this case we should say that the circle meets the axes in imaginary points. This form of statement, however, must be understood as simply another way of saying that the circle does not meet the axes.

26. Find the centre, radius, and intercepts on the axes, of the circle

$$x^2 + y^2 - 5x - 7y + 6 = 0. \qquad (1)$$

Writing the equation in the form of [15], we obtain

$$(x - \tfrac{5}{2})^2 + (y - \tfrac{7}{2})^2 = \tfrac{50}{4}.$$

Hence, $r = \frac{5}{2}\sqrt{2}$, and (a, b) is $(\frac{5}{2}, \frac{7}{2})$.

For $y = 0$ in (1), $x = 2$ and 3; hence the intercepts on the axis of x are 2 and 3.

For $x = 0$ in (1), $y = 1$ and 6; hence the intercepts on the axis of y are 1 and 6.

27. Find the centre, radius, and intercepts on the axes, of the circle

$$x^2 + y^2 - 12x - 4y + 15 = 0.$$

Writing this equation in the form of [15], we have

$$(x - 6)^2 + (y - 2)^2 = 25.$$

Hence $r = 5$, and (a, b) is $(6, 2)$.
For $y = 0$ in (1), $x = 6 \pm \sqrt{21} =$ the intercepts on the axis of x.
For $x = 0$, y is imaginary; hence the curve does not cut the axis of y.

28. Find the centre, radius, and intercepts on the axes, of the circle

$$x^2 + y^2 - 4x - 8y = 0.$$

Putting this equation in the form of [15], we have

$$(x - 2)^2 + (y - 4)^2 = 20.$$

Hence $r = 2\sqrt{5}$, and (a, b) is $(2, 4)$.

For $y = 0$ in (1), $x = 0$ and 4; that is, the curve cuts the axis of x at the origin and 4 to the right of it.

For $x = 0$ in (1), $y = 0$ and 8; that is, the curve cuts the axis of y at the origin and 8 above it.

29. Find the centre, radius, and intercepts on the axes, of the circle

$$x^2 + y^2 - 6x + 4y + 4 = 0. \qquad (1)$$

Putting this equation in the form of [15], we have

$$(x - 3)^2 + (y + 2)^2 = 3^2.$$

Hence $r = 3$, and the centre is $(3, -2)$.
For $y = 0$ in (1), $x = 3 \pm \sqrt{5} =$ intercepts on axis of x.
For $x = 0$ in (1), $y = -2 =$ intercept on axis of y.

30. Find the centre, radius, and intercepts on the axes, of the circle

$$x^2 + y^2 + 22x - 18y + 57 = 0. \qquad (1)$$

Putting this equation in the form of [15], we have

$$(x + 11)^2 + (y - 9)^2 = 145.$$

Hence the radius is $\sqrt{145}$, and the centre is $(-11, 9)$.

For $y = 0$ in (1), $x = -3$ and $-19 =$ intercepts on axis of x.

For $x = 0$ in (1), $y = 9 \pm 2\sqrt{6} =$ intercepts on axis of y.

31. Under what conditions will the circle

$$x^2 + y^2 + Dx + Ey + C = 0$$

(i.) touch the axis of x? (ii.) touch the axis of y? (iii.) not cut or touch either axis?

Putting this equation in the form of [15], we have

$$(x + \tfrac{1}{2}D)^2 + (y + \tfrac{1}{2}E)^2 = \tfrac{1}{4}(D^2 + E^2 - 4C).$$

Hence the centre is $(-\tfrac{1}{2}D, -\tfrac{1}{2}E)$, and $r = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}$.

(i.) Now the circle will touch the axis of x, when the ordinate of its centre equals numerically its radius; that is, when

$$\pm(-\tfrac{1}{2}E) = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}, \text{ or } D^2 = 4C.$$

(ii.) The circle will touch the axis of y, when the abscissa of its centre equals numerically its radius; that is, when

$$\pm(-\tfrac{1}{2}D) = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}, \text{ or } E^2 = 4C.$$

(iii.) The circle will not cut or touch either axis, when the co-ordinates of its centre are each numerically greater than its radius; that is, when

$$\pm(-\tfrac{1}{2}E) > \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}, \text{ or } D^2 < 4C,$$

and

$$\pm(-\tfrac{1}{2}D) > \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}, \text{ or } E^2 < 4C.$$

32. Show that the circle $x^2 + y^2 + 10x - 10y + 25 = 0$ touches the axes and lies entirely in the second quadrant. Write the equation so that it shall represent the same circle touching the axes and lying in the third quadrant.

Writing this equation in the form of [15], we have

$$(x + 5)^2 + (y - 5)^2 = 25.$$

Hence $r = 5$, and the centre is $(-5, 5)$.

The centre $(-5, 5)$ is in the second quadrant, and perpendiculars from it to the axes equal the radius 5; hence the axes are tangents, and the circle lies in the second quadrant.

If the circle whose radius is 5 lies in the third quadrant, and touches the axes, its centre must be $(-5, -5)$; hence its equation is

$$(x+5)^2 + (y+5)^2 = 25,$$

or

$$x^2 + y^2 + 10x + 10y + 25 = 0.$$

33. In what points does the straight line $3x + y = 25$ cut the circle $x^2 + y^2 = 65$?

Regarding these equations as simultaneous, and solving for x and y, we obtain $x = 7$, $y = 4$ and $x = 8$, $y = 1$; hence the points of intersection of their loci are $(7, 4)$ and $(8, 1)$.

34. Find the points common to the loci $x^2 + y^2 = 4$ and $y = 2x - 4$.

Regarding these equations as simultaneous, and solving for x and y, we obtain $x = \frac{6}{5}$, $y = -\frac{8}{5}$ and $x = 2$, $y = 0$; hence their common points are $(\frac{6}{5}, -\frac{8}{5})$ and $(2, 0)$.

35. The equation of a chord of the circle $x^2 + y^2 = 25$ is $y = 2x + 11$. Find its length.

Regarding these equations as simultaneous, and solving for x and y, we find the intersections of their loci, or the extremities of the chord, to be $(-4, 3)$ and $(-4\frac{4}{5}, 1\frac{2}{5})$.

By [1], the distance from $(-4, 3)$ to $(-4\frac{4}{5}, 1\frac{2}{5})$ is

$$\sqrt{(-4\tfrac{4}{5} + 4)^2 + (1\tfrac{2}{5} - 3)^2} = \tfrac{4}{5}\sqrt{5}.$$

36. The equation of a chord is $\frac{x}{a} + \frac{y}{b} = 1$; that of the circle is $x^2 + y^2 = r^2$. Find the length of the chord.

Putting the equation of the chord in the normal form, we obtain

$$\frac{b}{\sqrt{a^2 + b^2}}x + \frac{a}{\sqrt{a^2 + b^2}}y = \frac{ab}{\sqrt{a^2 + b^2}}. \qquad (1)$$

Hence the distance of the chord from the centre is $\frac{ab}{\sqrt{a^2 + b^2}}$.

But this distance, one-half of the chord, and the radius of the circle form a right triangle, of which r is the hypotenuse. Hence,

$$\text{chord} = 2\left(r^2 - \frac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}.$$

37. Find the equation of a line passing through the centre of

$$x^2 + y^2 - 6x - 8y = -21,$$

and perpendicular to $x + 2y = 4$.

Writing $x^2 + y^2 - 6x - 8y = -21$ in the form of [15], we have

$$(x-3)^2 + (y-4)^2 = 4.$$

Hence the centre of the circle is (3, 4).

The slope of $x + 2y = 4$ is $-\frac{1}{2}$; hence the slope of the perpendicular must be 2 (§ 45, Cor. 2).

By [5], the equation of the line through (3, 4) and having the slope 2 is

$$y - 4 = 2(x - 3), \text{ or } y = 2x - 2,$$

which is the required equation.

38. Find the equation of that chord of the circle $x^2 + y^2 = 130$ which passes through the point for which the abscissa is 9 and the ordinate negative, and which is parallel to the straight line $4x - 5y - 7 = 0$.

For $x = 9$, in $x^2 + y^2 = 130$, $y = \pm 7$; hence the line is to pass through the point $(9, -7)$.

The slope of $4x - 5y - 7 = 0$ is $\frac{4}{5}$; hence the slope of the required line is $\frac{4}{5}$.

By [5], the equation of the line through $(9, -7)$, and having the slope $\frac{4}{5}$, is

$$y + 7 = \tfrac{4}{5}(x - 9) \text{ or } 4x - 5y = 71.$$

39. What is the equation of the chord of the circle $x^2 + y^2 = 277$ which passes through $(3, -5)$ and is bisected at this point?

Let the equation of the chord be

$$x \cos \alpha + y \sin \alpha = p. \tag{1}$$

Since the perpendicular from the centre upon a chord bisects the chord, $(3, -5)$ must be the foot of the perpendicular p.

Hence $p = \sqrt{3^2 + (-5)^2} = \sqrt{34}$, $\cos \alpha = 3 \div \sqrt{34}$,

and $\sin \alpha = -5 \div \sqrt{34}$.

Substituting these values in (1), we obtain

$$\frac{3}{\sqrt{34}} x - \frac{5}{\sqrt{34}} y = \sqrt{34}, \text{ or } 3x - 5y = 34.$$

as the required equation.

Or, the equation of the line through (0, 0) and (3, −5) is

$$y = -\tfrac{5}{3}x. \qquad (2)$$

The line through (3, − 5) perpendicular to (2) is

$$y + 5 = \tfrac{3}{5}(x - 3), \text{ or } 3x - 5y = 34.$$

40. **Find the locus of the centre of a circle passing through the points (x_1, y_1) and (x_2, y_2).**

Let (x, y) be the centre of the circle.

Then, since (x_1, y_1) and (x_2, y_2) are points on the circle, they must be equally distant from the centre (x, y); hence we have

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2,$$

or

$$2(x_1 - x_2)x + 2(y_1 - y_2)y = (x_1^2 + y_1^2 - x_2^2 - y_2^2), \qquad (1)$$

which is the equation of the required locus.

Since (1) is a simple equation, the locus is a straight line.

The slope of (1) is $-\dfrac{x_2 - x_1}{y_2 - y_1}$, the slope of the line through (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$; hence (1) is perpendicular to the line through (x_1, y_1) and (x_2, y_2). The mid-point between (x_1, y_1) and (x_2, y_2) is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$, whose co-ordinates satisfy (1).

Hence (1) is perpendicular to the line through (x_1, y_1) and (x_2, y_2) at the point mid-way between them.

41. **What is the locus of the centres of all the circles which pass through the points (5, 3) and (− 7, − 6)?**

If in (1) of No. 40, we let (x_1, y_1) be (5, 3) and (x_2, y_2) be (− 7, − 6), we obtain, after simplifying,

$$8x + 6y + 17 = 0,$$

which is the equation of the required locus.

By No. 40 it is perpendicular to the line through (5, 3) and (− 7, − 6) at the point $(-1, -\tfrac{3}{2})$.

42. **Find the equation of a circle passing through the points (4, 0), (0, 4), (6, 4).**

The required equation will be of the form (§ 61),

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \qquad (1)$$

Since the locus of (1) passes through each of the three points (4, 0), (0, 4), and (6, 4), their co-ordinates must satisfy (1); hence, we have

$$16+8D+F=0, \tag{2}$$

$$16+8E+F=0, \tag{3}$$

and

$$36+16+12D+8E+F=0. \tag{4}$$

Solving these equations for D, E, and F, we obtain $D=-3$, $E=-3$, $F=8$. Substituting these values in (1), we obtain

$$x^2+y^2-6x-6y+8=0,$$

as the required equation.

43. Find the equation of a circle passing through the points (0, 0), (8, 0), (0, −6).

Substituting the co-ordinates of each of these points in the general equation of the circle, or (1) of § 61, we obtain

$$F=0, \tag{1}$$

$$64+16D+F=0, \tag{2}$$

$$36-12E+F=0. \tag{3}$$

Solving these equations for D, E, and F, we obtain

$$F=0,\ D=-4,\ E=3.$$

Substituting these values in the general equation, we have

$$x^2+y^2-8x+6y=0,$$

as the required equation.

44. Find the equation of a circle passing through the points (−6, −1), (0, 0), (0, −1).

Substituting the co-ordinates of each of these points in the general equation of the circle, or (1) of § 61, we obtain

$$36+1-12D-2E+F=0,$$

$$F=0,$$

$$1-2E+F=0.$$

Solving these equations for D, E, and F, we obtain

$$F=0,\ E=\tfrac{1}{2},\ D=3.$$

Substituting these values of D, E, F in the general equation of the circle, we obtain

$$x^2+y^2+6x+y=0.$$

45. Find the equation of a circle passing through the points $(0, 0)$, $(-8a, 0)$, $(0, 6a)$.

Substituting the co-ordinates of each of these points in the general equation of the circle, or (1) of § 61, we obtain

$$F = 0,$$
$$64a^2 - 16aD + F = 0,$$
$$36a^2 + 12aE + F = 0.$$

Whence $\quad F = 0,\ D = 4a,\ E = -3a.$

Hence $\quad x^2 + y^2 + 8ax - 6ay = 0$

is the required equation.

46. Find the equation of the circle passing through the points

$$(2, -3),\ (3, -4),\ (-2, -1).$$

Substituting the co-ordinates of each of these points in (1) of § 61, we obtain

$$4 + 9 + 4D - 6E + F = 0, \qquad (1)$$
$$9 + 16 + 6D - 8E + F = 0, \qquad (2)$$
$$4 + 1 - 4D - 2E + F = 0. \qquad (3)$$

Whence $\quad D = 4,\ E = 10,\ F = 31.$

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 + 8x + 20y + 31 = 0,$$

as the required equation.

47. Find the equation of the circle passing through the points

$$(1, 2),\ (1, 3),\ (2, 5).$$

Substituting the co-ordinates of each of these points in (1) of § 61, we obtain

$$1 + 4 + 2D + 4E + F = 0,$$
$$1 + 9 + 2D + 6E + F = 0,$$
$$4 + 25 + 4D + 10E + F = 0.$$

Whence $\quad D = -\frac{9}{2},\ E = -\frac{5}{2},\ F = 14.$

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 - 9x - 5y + 14 = 0.$$

48. Find the equation of the circle passing through $(10, 4)$ and $(17, -3)$, and radius $= 13$.

Substituting the value of r and the co-ordinate of each of these points in [15], we obtain

$$(10-a)^2+(4-b)^2=169, \qquad (1)$$

$$(17-a)^2+(-3-b)^2=169. \qquad (2)$$

Subtracting (2) from (1), we obtain

$$-189+14a+7-14b=0, \text{ or } a=b+13.$$

Substituting this value of a in (1) and solving for b, we obtain

$$b=9, \text{ and } -8.$$

Hence from (3) we have

$$a=22, \text{ and } 5.$$

Substituting these values in [15], we obtain

$$(x-22)^2+(y-9)^2=169,$$

and $$(x-5)^2+(y+8)^2=169,$$

as the required equations.

49. Find the equation of a circle passing through (3, 6), and touching the axes.

If the circle passes through (3, 6) and touches the axes, it must lie in the first quadrant, and we have $a=b=r$. Putting $x=3$, $y=6$, $a=r$, $b=r$ in [15], we obtain $(3-r)^2+(6-r)^2=r^2$.

Whence $a=b=r=15$ and 3.

Substituting these values in [15], we obtain

$$(x-15)^2+(y-15)^2=225,$$

and $$(x-3)^2+(y-3)^2=9,$$

as the required equations.

50. Find the equation of a circle touching each axis at the distance 4 from the origin.

If the circle touches each axis at the distance 4 from the origin, the centre must be (4, 4) and $r=4$.

Substituting these values in [15], we obtain

$$x^2+y^2-8x-8y+16=0,$$

as the required equation.

51. Find the equation of a circle touching each axis at the distance a from the origin.

If the circle touches each axis at the distance a from the origin, the centre must be (a, a) and $r = \pm a$ according as a is positive or negative.

Substituting these values in [15], we obtain

$$x^2 + y^2 \mp 2ax \mp 2ay + a^2 = 0.$$

52. Find the equation of a circle passing through the origin, and cutting the lengths a, b from the axes.

This circle passes through the three points $(0, 0)$, $(a, 0)$ and $(0, b)$. Substituting the co-ordinates of each of these points in (1) of § 61, we obtain

$$F = 0,$$
$$a^2 + 2aD + F = 0,$$
$$b^2 + 2bE + F = 0.$$

Whence $F = 0$, $D = -\frac{1}{2}a$, $E = -\frac{1}{2}b$.

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 - ax - by = 0,$$

as the required equation.

53. Find the equation of a circle passing through $(5, 6)$, and having its centre at the intersection of the lines $y = 7x - 3$, $4y - 3x = 13$.

Regarding $y = 7x - 3$ and $4y - 3x = 13$ as simultaneous, and solving, we find for the intersection of their loci the point $(1, 4)$. By [1], the distance from $(1, 4)$ to $(5, 6)$ is $\sqrt{(5 - 1)^2 + (6 - 4)^2}$, or $\sqrt{20}$. Hence $a = 1$, $b = 4$, and $r = \sqrt{20}$. Substituting these values in [15], we obtain

$$(x - 1)^2 + (y - 4)^2 = 20,$$

as the required equation.

54. Find the equation of a circle passing through $(10, 9)$ and $(5, 2 - 3\sqrt{6})$, and having its centre in the line $3x - 2y - 17 = 0$.

Since the circumference passes through $(10, 9)$ and $(5, 2 - 3\sqrt{6})$, its centre (a, b) must be equally distant from these points; hence, by [1], we have

$$(a - 10)^2 + (b - 9)^2 = (a - 5)^2 + (b - 2 + 3\sqrt{6})^2,$$

or

$$10a + (14 + 6\sqrt{6})b = 98 + 12\sqrt{6}. \qquad (1)$$

Moreover, since the centre (a, b) is on $3x - 2y - 17 = 0$, we have

$$3a - 2b = 17. \tag{2}$$

Solving (1) and (2), we obtain $a = 7$, $b = 2$.

Now the distance from (7, 2) to (10, 9) equals r; hence

$$r = \sqrt{3^2 + 7^2} = \sqrt{58}.$$

Substituting these values of a, b, r in [15], we have

$$x^2 + y^2 - 14x - 4y - 5 = 0,$$

as the required equation.

55. Find the equation of a circle passing through the origin, and cutting equal lengths a from the lines $x = y$, $x + y = 0$.

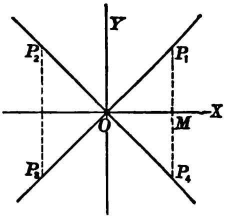

FIG. 51.

Let OX and OY be the axes, and P_3P_1, P_4P_2 the lines $y = x$ and $y = -x$, respectively, which bisect the angles between the axes. Let $OP_1 = OP_2 = OP_3 = OP_4 = a$; then, $OM = MP_1 = \frac{1}{2}a\sqrt{2}$. Hence, P_1 is $(\frac{1}{2}a\sqrt{2}, \frac{1}{2}a\sqrt{2})$, P_2 is $(-\frac{1}{2}a\sqrt{2}, \frac{1}{2}a\sqrt{2})$, P_3 is $(-\frac{1}{2}a\sqrt{2}, -\frac{1}{2}a\sqrt{2})$, and P_4 is $(\frac{1}{2}a\sqrt{2}, -\frac{1}{2}a\sqrt{2})$.

The circle passing through O, P_1, and P_4 has its centre on OX; hence its equation is of the form

$$x^2 + y^2 = 2rx. \tag{[17]}$$

Since P_1, $(\frac{1}{2}a\sqrt{2}, \frac{1}{2}a\sqrt{2})$, is on this circle, we must have

$$\tfrac{1}{2}a^2 + \tfrac{1}{2}a^2 = ra\sqrt{2}.$$

Whence $r = \frac{1}{2}a\sqrt{2}$. Substituting in [17], we have

$$x^2 + y^2 = ax\sqrt{2}.$$

The circle passing through O, P_1, P_2 is $x^2 + y^2 = ay\sqrt{2}$.

The circle passing through O, P_2, P_3 is $x^2 + y^2 = -ax\sqrt{2}$.

The circle passing through O, P_3, P_4 is $x^2 + y^2 = -ay\sqrt{2}$.

56. Find the equation of the circle circumscribing the triangle whose sides are the lines

$$y = 0,\ y = mx + b,\ \frac{x}{a} + \frac{y}{b} = 1.$$

The intersection of $y = 0$ and $y = mx + b$ is $\left(-\frac{b}{m}, 0\right)$;

that of $y = mx + b$ and $\frac{x}{a} + \frac{y}{b} = 1$ is $(0, b)$;

that of $\frac{x}{a} + \frac{y}{b} = 1$ and $y = 0$ is $(a, 0)$.

Substituting the co-ordinates of each of these vertices in (1) of § 61, we obtain

$$\frac{b^2}{m^2} - \frac{2b}{m}D + F = 0, \tag{1}$$

$$b^2 + 2bE + F = 0, \tag{2}$$

$$a^2 + 2aD + F = 0. \tag{3}$$

Solving these equations for D, E, and F, we obtain

$$D = \frac{b - am}{2m}, \quad E = \frac{a - mb}{2m}, \quad F = -\frac{ab}{m}.$$

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 + \frac{b - am}{m}x + \frac{a - mb}{m} - \frac{ab}{m} = 0,$$

or, $$m(x^2 + y^2) - ab = (am - b)x + (mb - a)y,$$

as the required equation.

57. Find the equation of a circle having for diameter the line joining $(0, 0)$ and (x_1, y_1).

The mid-point between $(0, 0)$ and (x_1, y_1) must be the centre; hence the centre (a, b) is $(\frac{1}{2}x_1, \frac{1}{2}y_1)$.

The radius must be one-half the distance between $(0, 0)$ and (x_1, y_1), or $r = \frac{1}{2}\sqrt{x_1^2 + y_1^2}$.

Substituting these values in [15], we have

$$(x - \tfrac{1}{2}x_1)^2 + (y - \tfrac{1}{2}y_1)^2 = \tfrac{1}{4}(x_1^2 + y_1^2), \text{ or } x^2 + y^2 = x_1x + y_1y,$$

as the required equation.

58. Find the equation of a circle having for diameter the line joining (x_1, y_1) and (x_2, y_2).

Here the centre is the mid-point between (x_1, y_1) and (x_2, y_2); hence

$$(a, b) \text{ is } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The radius is one-half the distance between (x_1, y_1) and (x_2, y_2); hence $r = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Substituting these values in [15], we obtain

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 = \tfrac{1}{4}(x_2 - x_1)^2 + \tfrac{1}{4}(y_2 - y_1)^2,$$

or $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0,$$

as the required equation.

59. Find the equation of a circle having for diameter the line joining the points where $y = mx$ meets $x^2 + y^2 = 2rx$.

Regarding these equations as simultaneous and solving, we find the common points of their loci to be

$$(0, 0) \text{ and } \left(\frac{2r}{1 + m^2}, \frac{2mr}{1 + m^2}\right).$$

Hence the centre of the circle is the mid-point between these points; or $\left(\frac{r}{1 + m^2}, \frac{mr}{1 + m^2}\right)$, and the radius is one-half the distance between them; or

$$r = \tfrac{1}{2}\sqrt{\frac{4r^2 + 4m^2r^2}{(1 + m^2)^2}} = \frac{r}{\sqrt{1 + m^2}}.$$

Substituting these values of a, b, and r in [15], we obtain

$$\left(x - \frac{r}{1 + m^2}\right)^2 + \left(y - \frac{mr}{1 + m^2}\right)^2 = \frac{r^2}{1 + m^2},$$

or $$(1 + m^2)(x^2 + y^2) - 2r(x + my) = 0,$$

as the required equation.

60. Find the equation of a circle having for diameter the common chord of the circles $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r^2$.

Regarding these equations as simultaneous and solving for x and y, we find the common points of their loci to be $(\frac{1}{2}a, \frac{1}{2}\sqrt{4r^2 - a^2})$ and $(\frac{1}{2}a, -\frac{1}{2}\sqrt{4r^2 - a^2})$.

Now the centre of the circle is the mid-point between these points or $(\frac{1}{2}a, 0)$, and the radius is one-half the distance between them, or

$$r = \tfrac{1}{2}\sqrt{4r^2 - a^2}.$$

Substituting these values of a, b, and r in [15], we have

$$(x - \tfrac{1}{2}a)^2 + y^2 = \tfrac{1}{4}(4r^2 - a^2),$$

or $$x^2 - ax + y^2 = r^2 - \tfrac{1}{4}a^2,$$

as the required equation.

Ex. 22. Page 81.

1. Explain the meaning of the double sign in equation [22].

The geometric interpretation of the double sign in [22] is, that to any circle there may be drawn two tangents having the same slope. If the circle be referred to its centre, the intercepts of these two tangents on the axis of y will be equal in length, but opposite in sign or direction.

2. Deduce the equations of the tangent and normal to the circle $x^2 + y^2 = r^2$, assuming that the normal passes through the centre.

If the normal to $x^2 + y^2 = r^2$ at the point (x_1, y_1) passes through the centre, its equation is

$$y = \frac{y_1}{x_1}x, \text{ or } x_1y - y_1x = 0. \tag{1}$$

By § 46 the equation of a line through (x_1, y_1), and perpendicular to (1), is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1) \text{ or } x_1x + y_1y = r^2. \tag{2}$$

Hence (2) is the equation of the tangent at (x_1, y_1).

3. Find the equations of the tangent and the normal passing through the point (4, 6) of the circle $x^2 + y^2 = 52$. Also the lengths of tangent, normal, subtangent, subnormal, and the portion of the tangent contained between the axes.

The equations of the tangent and normal to $x^2 + y^2 = r^2$ at the point (x_1, y_1) are respectively

$$x_1x + y_1y = r^2, \tag*{[18]}$$

and

$$y_1x - x_1y = 0. \tag*{[19]}$$

Here (x_1, y_1) is (4, 6) and $r^2 = 52$. Substituting these values in [18] and [19] we obtain

$$4x + 6y = 52, \text{ or } 2x + 3y = 26, \tag{1}$$

and

$$6x - 4y = 0, \text{ or } 3x - 2y = 0, \tag{2}$$

as the equations of the tangent and normal to $x^2 + y^2 = 52$ at the point (4, 6).

From Fig. 52 we have

$$\begin{aligned} \text{Subt} &= TM = OM - OT = x_1 - OT. && (A) \\ \text{Subn} &= MN = ON - OM = ON - x_1. && (B) \\ \text{Tan} &= PT = \sqrt{TM^2 + MP^2} \\ &= \sqrt{\text{Subt}^2 + y_1^2}. && (C) \\ \text{Nor} &= PN = \sqrt{MN^2 + MP^2} \\ &= \sqrt{\text{Subn}^2 + y_1^2}. && (D) \end{aligned}$$

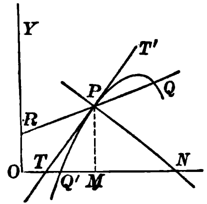

FIG. 52.

For $y = 0$ in (1) and (2), $x = 13$, and 0; hence we have $OT = 13$, and $ON = 0$; also $x_1 = 4$, and $y_1 = 6$. Substituting these values in (A) and (B), we obtain

$$\text{Subt} = 4 - 13 = -9. \tag{3}$$

$$\text{Subn} = 0 - 4 = -4. \tag{4}$$

The minus signs show that in this case the subtangent and subnormal extend in the direction opposite to that in the figure.

Substituting the values of subtangent, subnormal, and y_1 in (C) and (D), we obtain

$$\text{Tan} = \sqrt{81 + 36} = \sqrt{117}. = 3\sqrt{13}.$$

$$\text{Nor} = \sqrt{16 + 36} = \sqrt{52}. = 2\sqrt{13}.$$

For $x = 0$ in (1), $y = 8\frac{2}{3}$; hence the portion of the tangent between the axes is the distance between the points $(13, 0)$ and $(0, 8\frac{2}{3})$; that is, $\sqrt{13^2 + (8\frac{2}{3})^2}$, or $\frac{13}{3}\sqrt{13}$.

4. A straight line touches the circle $x^2 + y^2 = r^2$ in the point (x_1, y_1). Find the lengths of the subtangent, the subnormal, and the portion of the line contained between the axes.

The line is tangent to $x^2 + y^2 = r^2$ at (x_1, y_1), and its equation is

$$x_1x + y_1y = r^2. \tag{1}$$

For $y = 0$ in (1), $x = r^2 \div x_1 =$ intercept of tangent on the axis of x. Substituting this value of OT in (A) of No. 3, we obtain

$$\text{Subt} = x_1 - \frac{r^2}{x_1} = \frac{x_1^2 - r^2}{x_1}.$$

The equation of the normal to $x^2 + y^2 = r^2$ at (x_1, y_1) is

$$y_1x - x_1y = 0. \tag{2}$$

For $y = 0$ in (2), $x = 0 =$ intercept of normal on the axis of x. Substituting this value of ON in (B) of No. 3, we have

$$\text{Subn} = 0 - x_1 = -x_1.$$

For $x = 0$ in (1), $y = r^2 \div y_1 =$ intercept of tangent on the axis of y. The portion of the line between the axes forms with its intercepts a right triangle, and hence equals

$$\sqrt{\frac{r^4}{x_1^2} + \frac{r^4}{y_1^2}} = r^2\sqrt{\frac{1}{x_1^2} + \frac{1}{y_1^2}} = r^2\sqrt{\frac{y_1^2 + x_1^2}{x_1^2 y_1^2}} = \frac{r^3}{x_1 y_1}.$$

5. What is the equation of a tangent to the circle $x^2 + y^2 = 250$ at the point whose abscissa is 9 and ordinate negative?

Here $r^2 = 250$, and for $x = 9$ in $x^2 + y^2 = 250$, $y = \pm 13$; hence (x_1, y_1) is $(9, -13)$. Substituting these values in [18] we obtain

$$9x - 13y = 250$$

as the required equation.

6. Find the equations of tangents to $x^2 + y^2 = 10$ at the points whose common abscissa $= 1$.

Here $r^2 = 10$, and for $x = 1$ in $x^2 + y^2 = 10$, $y = \pm 3$; hence (x_1, y_1) is $(1, 3)$ and $(1, -3)$. Substituting these values in [18], we obtain

$$x + 3y = 10, \text{ and } x - 3y = 10,$$

as the required equations.

7. Tangents are drawn through the points of the circle $x^2 + y^2 = 25$ which have abscissas numerically equal to 3. Prove that these tangents enclose a rhombus, and find its area.

Here $r^2 = 25$, and for $x = 3$ in $x^2 + y^2 = 25$, $y = \pm 4$, for $x = -3$, $y = \pm 4$; hence (x_1, y_1) is $(3, 4)$, $(3, -4)$, $(-3, 4)$, and $(-3, -4)$. Substituting these values in [18] in succession, we obtain

$$3x + 4y = 25, \tag{1}$$
$$3x - 4y = 25, \tag{2}$$
$$-3x + 4y = 25, \tag{3}$$
$$-3x - 4y = 25, \tag{4}$$

as the equations of the four tangents.

Now lines (1) and (4), having the same slope, are parallel; for like reason lines (2) and (3) are parallel. Hence the figure is a parallelogram. Either of these lines is at the distance 5 from the origin; hence the distance between either two opposite sides of the parallelogram is 10. Let a and b be two consecutive sides; then area $= 10a$ or $10b$. Hence $a = b$, and the figure is a rhombus.

Regarding (1) and (2) as simultaneous, and also (1) and (3), we find that two consecutive vertices are the points $(\frac{25}{3}, 0)$ and $(0, \frac{25}{4})$. By [1] the distance between these vertices, or the length of a side, is

$$\sqrt{(-\tfrac{25}{3})^2 + (\tfrac{25}{4})^2} = \tfrac{125}{12}.$$

Hence, $\quad$ $\text{area} = \frac{125}{12} \times 10 = 104\frac{1}{6}.$

8. The subtangent for a certain point of a circle is $5\frac{1}{2}$; the subnormal is 3. What is the equation of the circle?

From No. 4 we have

$$\text{Subt} = \frac{x_1^2 - r^2}{x_1}, \quad \text{Subn} = -x_1.$$

Hence we have

$$\text{Subn} = -x_1 = 3, \text{ or } x_1 = -3.$$

Substituting the value of x_1 in the value of the subtangent, we obtain

$$\text{Subt} = \frac{9 - r^2}{-3} = 5\tfrac{1}{2}, \text{ or } r^2 = 25\tfrac{1}{2}.$$

Hence the equation of the circle is

$$x^2 + y^2 = 25\tfrac{1}{2}.$$

9. Find the equation of a straight line touching $x^2 + y^2 = 232$ at the point whose abscissa $= 14$.

Here $r^2 = 232$, and for $x = 14$ in $x^2 + y^2 = 232$, $y = \pm 6$; hence the point of contact (x_1, y_1) is $(14, 6)$ or $(14, -6)$.

Substituting these values in succession in [18], we obtain

$$14x + 6y = 232, \text{ and } 14x - 6y = 232,$$

as the required equations of the tangents.

10. Find the equation of a straight line touching

$$(x - 2)^2 + (y - 3)^2 = 10$$

at the point $(5, 4)$.

The equation of a tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at (x_1, y_1) is

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2. \qquad [20]$$

Here the centre (a, b) is $(2, 3)$, the point of contract (x_1, y_1) is $(5, 4)$ and $r^2 = 10$. Substituting these values in [20], we obtain

$$(5 - 2)(x - 2) + (4 - 3)(y - 3) = 10,$$

or $\quad$ $3x + y = 19,$

as the required equation.

11. Find the equation of a straight line touching

$$x^2 + y^2 - 3x - 4y = 0$$

at the origin.

This equation may be written in the form

$$(x - \tfrac{3}{2})^2 + (y - 2)^2 = \tfrac{25}{4}. \tag{1}$$

The equation of the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point (x_1, y_1) is

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2. \qquad [20]$$

Here (a, b) is $(\tfrac{3}{2}, 2)$, (x_1, y_1) is $(0, 0)$ and $r^2 = \tfrac{25}{4}$. Substituting these values in [20], we obtain

$$(-\tfrac{3}{2})(x - \tfrac{3}{2}) - 2(y - 2) = \tfrac{25}{4}, \text{ or } 3x + 4y = 0,$$

as the required equation.

12. Find the equation of a line touching

$$x^2 + y^2 - 14x - 4y - 5 = 0$$

at the point whose abscissa is equal to 10.

For $x = 10$ in $x^2 + y^2 - 14x - 4y - 5 = 0$, $y = 9$, and -5; hence (x_1, y_1) is $(10, 9)$ and $(10, -5)$.

The given equation may be written in the form

$$(x - 7)^2 + (y - 2)^2 = 58;$$

hence (a, b) is $(7, 2)$ and $r^2 = 58$.

Substituting these values in [20], we obtain

$$(10 - 7)(x - 7) + (9 - 2)(y - 2) = 58, \text{ or } 3x + 7y = 93,$$

and

$$(10 - 7)(x - 7) + (-5 - 2)(y - 2) = 58, \text{ or } 3x - 7y = 65,$$

as the required equations.

13. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also passing through the point of contact $(r, 0)$?

The equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point of contact $(x_1\ y_1)$ is

$$x_1x + y_1y = r^2. \qquad [18]$$

Here (x_1, y_1) is $(r, 0)$. Substituting in [18], we obtain

$$x = r$$

as the required equation.

14. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also parallel to the line $Ax + By + C = 0$?

I. *Solution.* The slope of (1) is $-\frac{A}{B}$, and that of [18] is $-\frac{x_1}{y_1}$. Hence if the tangent is parallel to (1), we must have

$$-\frac{A}{B} = -\frac{x_1}{y_1}. \tag{1}$$

Again, since (x_1, y_1) is on the circle, we have

$$x_1^2 + y_1^2 = r^2. \tag{2}$$

Solving (1) and (2), we obtain

$$x_1 = \frac{\pm rA}{\sqrt{A^2 + B^2}}, \quad y_1 = \frac{\pm rB}{\sqrt{A^2 + B^2}}.$$

Substituting these values of x_1 and y_1 in [18], we obtain

$$Ax + By = \pm r\sqrt{A^2 + B^2} \tag{3}$$

as the equation required.

II. *Solution.* Substituting the slope $-\frac{A}{B}$ for m in

$$y = mx \pm r\sqrt{1 + m^2}, \tag{22}$$

we obtain after reduction

$$Ax + By = \pm r\sqrt{A^2 + B^2}.$$

15. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also perpendicular to the line $Ax + By + C = 0$? (1)

If the tangent is perpendicular to $Ax + By + C = 0$, it will be parallel to $Ax - By + C = 0$. Substituting $-B$ for B in (3) of No. 14, we obtain as the required equation

$$Ax - By = \pm r\sqrt{A^2 + B^2}. \tag{2}$$

Or, since the tangent is perpendicular to $Ax + By + C = 0$, its slope is $B \div A$. Substituting this value for m in [22], we obtain after reduction

$$Ax - By = \pm r\sqrt{A^2 + B^2}.$$

16. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also making the angle 45° with the axis of x?

Here $m = 1$. Substituting this value of m in [22], we obtain as the required equation

$$y = x \pm r\sqrt{2}. \qquad (1)$$

Or, since $\gamma = 45°$, $m = \tan\gamma = 1$. Hence we have

$$-\frac{x_1}{y_1} = 1, \text{ and } x_1^2 + y_1^2 = r^2.$$

Solving these equations, and substituting in [18], we obtain (1) as the required equation.

17. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also passing through the exterior point $(h, 0)$?

If $x_1x + y_1y = r^2$ passes through $(h, 0)$, we have

$$hx_1 = r^2, \text{ or } x_1 = r^2 \div h.$$

Since (x_1, y_1) is on $x^2 + y^2 = r^2$, we have

$$x_1^2 + y_1^2 = r^2. \qquad (2)$$

From (1) and (2) we obtain

$$y_1 = \pm\frac{r}{h}\sqrt{h^2 - r^2}.$$

Hence the points of contact are

$$\left(\frac{r^2}{h}, \frac{r}{h}\sqrt{h^2 - r^2}\right) \text{ and } \left(\frac{r^2}{h}, -\frac{r}{h}\sqrt{h^2 - r^2}\right).$$

Substituting these values of (x_1, y_1) in [18], we obtain

$$\frac{r^2}{h}x \pm \frac{r}{h}\sqrt{h^2 - r^2}\,y = r^2, \text{ or } r^2(x - h)^2 = (h^2 - r^2)y^2,$$

as the equation required.

The use of [22] instead of [18] would give a shorter solution.

18. What is the equation of a straight line touching the circle $x^2 + y^2 = r^2$, and also forming a triangle of area r^2 with the axes?

For $x = 0$ in $x_1x + y_1y = r^2$, $y = r^2 \div y_1$ = intercept on axes of y.

For $y = 0$, $x = r^2 \div x_1$ = intercept on axis of x.

Hence the area of the triangle which a tangent to the circle forms with the axes equals $r^4 \div 2x_1y_1$; hence

$$r^2 = \frac{r^4}{2x_1y_1}, \text{ or } r^2 = 2x_1y_1. \qquad (1)$$

But we have also $x_1^2 + y_1^2 = r^2$. (2)

Solving (1) and (2), we obtain

$$x_1 = y_1 = \tfrac{1}{2}r\sqrt{2}.$$

Substituting these values in [18], we obtain

$$x + y = \sqrt{2}\,r$$

as the required equation.

19. Find the equations of the tangents drawn from the point (10, 5) to the circle $x^2 + y^2 = 100$.

By [18] the equation of the tangent to $x^2 + y^2 = 100$ at (x_1, y_1) is

$$x_1x + y_1y = 100. \qquad (1)$$

Since (10, 5) is on (1), we must have

$$10\,x_1 + 5\,y_1 = 100, \text{ or } 2\,x_1 + y_1 = 20. \qquad (2)$$

Since (x_1, y_1) is on $x^2 + y^2 = 100$, we have

$$x_1^2 + y_1^2 = 100.$$

Solving (2) and (3), we obtain

$$x_1 = 10,\ y_1 = 0, \text{ and } x_1 = 6,\ y_1 = 8.$$

Substituting these values in (1), we obtain

$$x = 10, \text{ and } 3\,x + 4\,y = 50,$$

as the equations required.

20. Find the equations of tangents to the circle

$$x^2 + y^2 + 10\,x - 6\,y - 2 = 0$$

and parallel to the line $y = 2\,x - 7$.

Since the tangents are to be parallel to $2\,x - y = 7$, their equations will be of the form $2\,x - y = C$.

The equation $x^2 + y^2 + 10\,x - 6\,y - 2 = 0$ may be written in the form

$$(x + 5)^2 + (y - 3)^2 = 36. \qquad (2)$$

Thus the centre is $(-5, 3)$ and $r = 6$. Now the distance of (1) from $(-5, 3)$ is $\dfrac{-10 - 3 - C}{\sqrt{5}}$. But if (1) is a tangent, this distance must be ± 6; hence we have

$$\frac{-13 - C}{\sqrt{5}} = \pm 6, \text{ or } C = -13 \mp 6\sqrt{5}.$$

Substituting these values of C in (1), we obtain

$$y = 2\,x + 13 \pm 6\sqrt{5}$$

as the required equation.

21. Find the lengths of subtangent and subnormal in the circle $x^2 + y^2 - 14x - 4y = 5$ for the point (10, 9).

The equation of the circle may be written in the form

$$(x-7)^2 + (y-2)^2 = 58. \qquad (1)$$

By [20] the equation of the tangent to (1) at (10, 9) is

$$(10-7)(x-7) + (9-2)(y-2) = 58, \text{ or } 3x + 7y = 93. \qquad (2)$$

For $y = 0$ in (2), $x = 31 =$ intercept on axis of x.

Substituting this value and that of x_1 in (A) of No. 3, we have

$$\text{Subt} = x_1 - OT = 10 - 31 = -21.$$

By [21] the equation of the normal to (1) at (10, 9) is

$$(9-2)(x-10) - (10-7)(y-9) = 0, \text{ or } 7x - 3y = 43. \qquad (3)$$

For $y = 0$ in (3), $x = 6\frac{1}{7} =$ intercept of normal on axis of x.

Substituting this value, and that of x_1 in (B) of No. 3, we have

$$\text{Subn} = ON - x_1 = 6\tfrac{1}{7} - 10 = -3\tfrac{6}{7}.$$

The minus signs in the values of the subtangent and subnormal show that the subtangent extends from the foot of the ordinate to the right, and the subnormal from the same point to the left.

22. What is the equation of the circle (centre at origin) which is touched by the straight line $x \cos \alpha + y \sin \alpha = p$? What are the co-ordinates of the point of contact?

The distance of this straight line from the origin is p.

Hence the radius of the circle is p, and its equation is

$$x^2 + y^2 = p^2. \qquad (1)$$

Multiplying both members of $x \cos \alpha + y \sin \alpha = p$ by p, we have

$$xp \cos \alpha + yp \sin \alpha = p^2. \qquad (2)$$

The equation of the tangent to (1) at (x_1, y_1) is

$$x_1 x + y_1 y = p^2. \qquad (3)$$

Comparing (2) and (3), we obtain

$$x_1 = p \cos \alpha, \quad y_1 = p \sin \alpha.$$

23. When will the line $Ax + By - C = 0$ touch the circle $x^2 + y^2 = r^2$? the circle $(x-a)^2 + (y-b)^2 = r^2$?

The distance of $Ax + By - C = 0$ from the origin is $C \div \sqrt{A^2 + B^2}$.

Hence when $C \div \sqrt{A^2 + B^2} = r$, or $C = r\sqrt{A^2 + B^2}$, the line will touch the circle $x^2 + y^2 = r^2$.

The distance of a tangent to $(x-a)^2+(y-b)^2=r^2$ from (a, b) is numerically r, and the distance of $Ax+By-C=0$ from (a, b) is

$$\frac{Aa+Bb-C}{\sqrt{A^2+B^2}}.$$

Hence when the line is tangent to the circle, we must have

$$\frac{Aa+Bb-C}{\sqrt{A^2+B^2}}=\pm r.$$

24. Find the equation of a straight line touching $x^2+y^2=ax+by$ and passing through the origin.

The equation of the circle can be written in the form

$$(x-\tfrac{1}{2}a)^2+(y-\tfrac{1}{2}b)^2=\tfrac{1}{4}(a^2+b^2). \qquad (1)$$

By [20] the equation of a tangent to (1) at (x_1, y_1) is

$$(x_1-\tfrac{1}{2}a)(x-\tfrac{1}{2}a)+(y_1-\tfrac{1}{2}b)(y-\tfrac{1}{2}b)=\tfrac{1}{4}(a^2+b^2). \qquad (2)$$

Since this tangent passes through (0, 0), we have

$$-\tfrac{1}{2}a(x_1-\tfrac{1}{2}a)-\tfrac{1}{2}b(y_1-\tfrac{1}{2}b)=\tfrac{1}{4}(a^2+b^2),$$

or

$$ax_1+by_1=0. \qquad (3)$$

Since (x_1, y_1) is on the given circle, we have

$$x_1^2+y_1^2=ax_1+by_1. \qquad (4)$$

Combining (3) and (4), we find

$$x_1=y_1=0.$$

Substituting these values in (2), we obtain

$$ax+by=0$$

as the required equation.

Or, since the line passes through the origin, its equation is

$$y=mx. \qquad (5)$$

Putting the distances of $y=mx$ from the centre $(\tfrac{1}{2}a, \tfrac{1}{2}b)$ equal to $\pm r$, or $\pm\tfrac{1}{2}\sqrt{a^2+b^2}$, we can find the value of m.

25. Prove that the circle $x^2+y^2+ax+by=0$ and the straight line $ax+by+a^2+b^2=0$ touch, and find the point of contact.

Regarding these equations as simultaneous, and solving for x and y, we find that the only common point of their loci is $(-a, -b)$; hence the straight line is a tangent to the circle at this point.

Or, by putting the first equation in the form

$$(x+\tfrac{1}{2}a)^2+(y+\tfrac{1}{2}b)^2=\tfrac{1}{4}(a^2+b^2),$$

we learn that the centre of the circle is

$$(-\tfrac{1}{2}a,\ -\tfrac{1}{2}b),\text{ and } r=\tfrac{1}{2}\sqrt{a^2+b^2}.$$

But the distance of $ax+by+a^2+b^2=0$ from $(-\frac{1}{2}a, -\frac{1}{2}b)$ is

$$-\tfrac{1}{2}\sqrt{a^2+b^2};$$

hence the line is tangent to the circle.

The slope of this tangent is $-(a \div b)$; hence the slope of the normal to the point of contact is $b \div a$, and the equation of the normal is

$$ay-bx=0.$$

The point of contact is found by determining the point of intersection of tangent and normal.

26. Prove that the circle $x^2+y^2-2ax-2by+b^2=0$ and the straight line $x=2a$ touch, and find the point of contact.

Regarding these equations as simultaneous, and solving for x and y, we find that the only common point of their loci is $(2a, b)$; hence the line is tangent to the circle at this point.

27. Prove that the circle $x^2+y^2=ax+by$ and the straight line $ax-by+b^2=0$ touch, and find the point of contact.

Regarding these equations as simultaneous, and solving for x and y, we find that the only common point of their loci is $(0, b)$; hence the straight line is tangent to the circle at this point.

28. What is the equation of the circle (centre at origin) which touches the line $y=3x-5$?

Reducing $y=3x-5$ to the normal form, we have

$$\frac{3x}{\sqrt{10}}-\frac{y}{\sqrt{10}}=\frac{5}{\sqrt{10}}=\tfrac{1}{2}\sqrt{10}.$$

Hence the distance of this line from the origin is $\frac{1}{2}\sqrt{10}$, and the line is therefore tangent to the circle

$$x^2+y^2=(\tfrac{1}{2}\sqrt{10})^2=2\tfrac{1}{2}.$$

29. What must be the value of m in order that the line $y = mx + 10$ may touch the circle $x^2 + y^2 = 100$? Show that we get the same answer for the line $y = mx - 10$, and explain the reason.

The distance of $y = mx + 10$ from the origin is $\frac{10}{\sqrt{1 + m^2}}$.

Hence that $y = mx + 10$ may touch the circle $x^2 + y^2 = 10^2$, we must have $\frac{10}{\sqrt{1 + m^2}} = 10$, or $m = 0$.

The distance of $y = mx - 10$ from the origin is also $\frac{10}{\sqrt{1 + m^2}}$.

Hence when this line is tangent to $x^2 + y^2 = 10^2$, we have $m = 0$. The lines $y = mx + 10$ and $y = mx - 10$ are the same distance from the origin, but on opposite sides of it.

30. Determine the value of c in order that the line $3x - 4y + c = 0$ may touch the circle $x^2 + y^2 - 8x + 12y - 44 = 0$. Explain the double answer.

The equation of the circle may be written in the form

$$(x - 4)^2 + (y + 6)^2 = 96. \qquad (1)$$

Hence its centre is $(4, -6)$ and $r = 4\sqrt{6}$.
The distance of $-3x + 4y - c = 0$ from the centre $(4, -6)$ is

$$\frac{-12 - 24 - c}{5}, \text{ or } \frac{-36 - c}{5}.$$

Hence that this line may be a tangent to (1), we must have

$$\frac{-36 - c}{5} = \pm 4\sqrt{6}, \text{ or } c = -36 \mp 20\sqrt{6}.$$

The double answer shows that two tangents can be drawn having the slope $\frac{3}{4}$.

31. What is the equation of the circle having for centre the point (5, 3) and touching the line $3x + 2y - 10 = 0$?

The distance of the line $3x + 2y - 10 = 0$ from (5, 3) is, by [12],

$$\frac{15 + 6 - 10}{\sqrt{13}}, \text{ or } \frac{11}{\sqrt{13}}.$$

Hence the radius of the required circle is $\frac{11}{\sqrt{13}}$. Its centre is $(5, 3)$, and therefore its equation is $(x-5)^2+(y-3)^2=\frac{121}{13}$.

32. What is the equation of a circle whose radius $= 10$, and which touches the line $4x+3y-70=0$ in the point $(10, 10)$?

Let (a, b) be its centre; then, since $r = 10$, its equation is

$$(x-a)^2+(y-b)^2=100. \tag{1}$$

Since (1) passes through $(10, 10)$, we have

$$(10-a)^2+(10-b)^2=100. \tag{2}$$

The distance of $4x+3y-70=0$ from (a, b) is

$$\frac{4a+3b-70}{5}.$$

But this distance is ± 10; hence we have

$$\frac{4a+3b-70}{5}=\pm 10. \tag{3}$$

Solving (2) and (3), we find (a, b) to be $(2, 4)$ or $(18, 16)$.

Substituting these values in (1), we obtain

$$(x-2)^2+(y-4)^2=100,$$

and $$(x-18)^2+(y-16)^2=100,$$

as the equations required.

33. A circle touching the line $4x+3y+3=0$ in the point $(-3, 3)$ passes through the point $(5, 9)$. What is its equation?

Since the circle passes through $(-3, 3)$ and $(5, 9)$, its centre (a, b) must be equally distant from them, and by [1] we have

$$(a+3)^2+(b-3)^2=(a-5)^2+(b-9)^2, \text{ or } 4a+3b=22. \tag{1}$$

Since the circle is tangent to $4x+3y+3=0$ at $(-3, 3)$, its centre is on the perpendicular to this line at $(-3, 3)$, whose equation is

$$y-3=\tfrac{3}{4}(x+3), \text{ or } 4y-3x=21. \tag{2}$$

Since (a, b) is on (2), we have

$$4b-3a=21. \tag{3}$$

Solving (1) and (3), we find $a=1$, $b=6$.

The distance from the centre $(1, 6)$ to the point $(5, 9)$ or $(-3, 3)$ is, by [1], equal to 5; hence $r=5$. Substituting these values in [15], we have $(x-1)^2+(y-6)^2=25$, as the required equation.

34. Under what condition will the line $\frac{x}{a}+\frac{y}{b}=1$ touch the circle $x^2+y^2=r^2$?

Reducing $\frac{x}{a}+\frac{y}{b}=1$ to the normal form, we have

$$\frac{bx+ay}{\sqrt{a^2+b^2}}=\frac{ab}{\sqrt{a^2+b^2}}. \qquad (1)$$

Putting the distance of (1) from the origin equal to r, we have

$$r=\frac{ab}{\sqrt{a^2+b^2}}, \text{ or } \frac{1}{r^2}=\frac{a^2+b^2}{a^2b^2}=\frac{1}{b^2}+\frac{1}{a^2},$$

as the required condition.

35. What is the equation of the circle inscribed in the triangle whose sides are $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}=1$?

The bisector of the angle between $x=0$ and $y=0$ is

$$y=x. \qquad (1)$$

The bisector of the angle between $y=0$ and $bx+ay=ab$ is

$$-\frac{bx+ay-ab}{\sqrt{a^2+b^2}}=y. \qquad (2)$$

Solving equations (1) and (2) for x and y, we obtain

$$x=y=\frac{ab}{a+b+\sqrt{a^2+b^2}}.$$

Hence the centre is

$$\left(\frac{ab}{a+b+\sqrt{a^2+b^2}}, \frac{ab}{a+b+\sqrt{a^2+b^2}}\right), \text{ and } r=\frac{ab}{a+b+\sqrt{a^2+b^2}},$$

and the equation of the circle is

$$\left(x-\frac{ab}{a+b+\sqrt{a^2+b^2}}\right)^2+\left(y-\frac{ab}{a+b+\sqrt{a^2+b^2}}\right)^2=\frac{a^2b^2}{(a+b+\sqrt{a^2+b^2})^2},$$

or

$$(x^2+y^2)(a+b+\sqrt{a^2+b^2})^2-2ab(a+b+\sqrt{a^2+b^2})(x+y)+a^2b^2=0.$$

36. Two circles touch each other when the distance between their centres is equal to the sum or the difference of their radii. Prove that the circles

$$x^2+y^2=(r+a)^2, \qquad (x-a)^2+y^2=r^2,$$

touch each other, and find the equation of the common tangent.

The centre of $x^2 + y^2 = (r + a)^2$ is $(0, 0)$, and its radius is $r + a$.

The centre of $(x - a)^2 + y^2 = r^2$ is $(a, 0)$, and its radius is r. Hence the distance between the centres $(0, 0)$ and $(a, 0)$ is equal to the difference of their radii $r + a$ and r, and the smaller circle is tangent to the other internally. Regarding their equations as simultaneous, we find the common point of these circles to be $(r + a, 0)$. Now the tangent to $x^2 + y^2 = (r + a)^2$ at the point $(r + a, 0)$ is

$$(r + a)x = (r + a)^2, \text{ or } x = r + a.$$

which is the required common tangent.

37. Two circles touch each other when the length of their common chord $= 0$. Find the length of the common chord of

$$(x - a)^2 + (y - b)^2 = r^2, \qquad (x - b)^2 + (y - a)^2 = r^2,$$

and hence prove that the two circles touch each other when

$$(a - b)^2 = 2r^2.$$

Subtracting the first equation from the second, we obtain as the equation of their common chord

$$x - y = 0. \tag{1}$$

The distance of (1) from the centre (a, b) is $\frac{a - b}{\sqrt{2}}$. But this distance and half the chord are the sides, and the radius r is the hypotenuse of a right triangle; hence

$$\text{Chord} = 2\sqrt{r^2 - \frac{(a - b)^2}{2}} = [4r^2 - 2(a - b)^2]^{\frac{1}{2}}.$$

The length of this common chord becomes zero, and hence the circles are tangent when $2r^2 = (a - b)^2$.

We might solve No. 36 as follows:

The equation of their common chord is $x = r + a$, which, being at the distance $r + a$ from the origin, is tangent to $x^2 + y^2 = (2 + a)^2$.

Ex. 23. Page 84.

1. Find the radius and centre of the circle

$$3x^2 - 6x + 3y^2 + 9y - 12 = 0.$$

Dividing by 3, this equation may be written in the form

$$(x - 1)^2 + (y + \tfrac{3}{2})^2 = \tfrac{29}{4}. \tag{1}$$

Hence $r = \frac{1}{2}\sqrt{29}$, and the centre is $(1, -\frac{3}{2})$.

2. Find the radius and centre of the circle

$$7x^2 + 3y^2 - 4y - (1 - 2x)^2 = 0.$$

This equation may be written in the form

$$(x + \tfrac{2}{3})^2 + (y - \tfrac{2}{3})^2 = \tfrac{11}{9}.$$

Hence $r = \frac{1}{3}\sqrt{11}$, and the centre is $(-\frac{2}{3}, \frac{2}{3})$.

3. Find the radius and centre of the circle $y(y - 5) = x(3 - x)$.

This equation may be written in the form

$$(x - \tfrac{3}{2})^2 + (y - \tfrac{5}{2})^2 = \tfrac{17}{2}.$$

Hence $r = \frac{1}{2}\sqrt{34}$, and the centre is $(\frac{3}{2}, \frac{5}{2})$.

4. Find the radius and centre of the circle

$$\sqrt{1 + a^2}\,(x^2 + y^2) = 2b\,(x + ay).$$

This equation may be written in the form

$$\left(x - \frac{b}{\sqrt{1 + a^2}}\right)^2 + \left(y - \frac{ab}{\sqrt{1 + a^2}}\right)^2 = b^2.$$

Hence $r = b$, and the centre is $\left(\frac{b}{\sqrt{1 + a^2}}, \frac{ab}{\sqrt{1 + a^2}}\right)$.

5. Find the equation of a circle: centre (0, 0), radius = 9.

Here $r = 9$, and (a, b) is (0, 0). Substituting these values in [15], we obtain

$$x^2 + y^2 = 81$$

as the required equation.

6. Find the equation of a circle: centre (7, 0), radius = 3.

Here $r = 3$, and (a, b) is (7, 0). Substituting these values in [15], we obtain

$$(x - 7)^2 + y^2 = 9$$

as the required equation.

7. Find the equation of a circle: centre (−2, 5), radius = 10.

Here $r = 10$, and (a, b) is (−2, 5). Substituting these values in [15], we obtain

$$(x + 2)^2 + (y - 5)^2 = 100$$

as the required equation.

8. Find the equation of a circle: centre $(3a, 4a)$, radius $= 5a$.

Here $r = 5a$, and (a, b) is $(3a, 4a)$. Substituting these values in [15], we obtain

$$(x - 3a)^2 + (y - 4a)^2 = 25a^2,$$

or $$x^2 + y^2 - 2a(3x + 4y) = 0$$

as the required equation.

9. Find the equation of a circle: centre $(b + c, b - c)$, radius $= c$.

Here $r = c$, and (a, b) is $(b + c, b - c)$. Substituting these values in [15], we obtain

$$(x - b - c)^2 + (y - b + c)^2 = c^2,$$

or $$x^2 + y^2 + 2b^2 + c^2 - 2[(b + c)x + (b - c)y] = 0$$

as the required equation.

10. Find the equation of a circle passing through $(a, 0)$, $(0, b)$, $(2a, 2b)$.

By § 61 the required equation will be of the form

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \qquad (a)$$

Since (a) passes through each of the three points, we have

$$a^2 + 2Da + F = 0, \qquad (1)$$

$$b^2 + 2Eb + F = 0, \qquad (2)$$

and $$4a^2 + 4b^2 + 4Da + 4Eb + F = 0. \qquad (3)$$

Solving these equations for D, E, and F, we obtain

$$D = -\frac{5a^2 + 2b^2}{6a},\quad E = -\frac{5b^2 + 2a^2}{6b},\quad F = \tfrac{2}{3}(a^2 + b^2).$$

Substituting these values of D, E, F, in (1), we obtain

$$x^2 + y^2 - \frac{5a^2 + 2b^2}{3a}x - \frac{5b^2 + 2a^2}{3b}y + \tfrac{2}{3}(a^2 + b^2) = 0,$$

or $$3ab(x^2 + y^2) - (5a^2 + 2b^2)bx - (5b^2 + 2a^2)ay + 2ab(a^2 + b^2) = 0$$

as the required equation.

11. Find the equation of a circle passing through $(0, 0)$, $(0, 12)$, $(5, 0)$.

Substituting the co-ordinates of each point in (1) of § 61, we obtain

$$F = 0, \quad (1)$$

$$144 + 24E + F = 0, \quad (2)$$

$$25 + 10D + F = 0. \quad (3)$$

Solving these equations for D, E, and F, we obtain

$$F = 0,\ E = -6,\ D = -\tfrac{5}{2}.$$

Substituting these values in (1) of § 61, we have

$$x^2 + y^2 - 5x - 12y = 0$$

as the required equation.

12. Find the equation of a circle passing through (10, 9), (4, −5), (0, 5).

Substituting the co-ordinates of each point in (1) of § 61, we obtain

$$100 + 81 + 20D + 18E + F = 0, \quad (1)$$

$$16 + 25 + 8D - 10E + F = 0, \quad (2)$$

and

$$25 + 10E + F = 0. \quad (3)$$

Solving these equations for D, E, and F, we obtain

$$D = -7,\ E = -2,\ F = -5.$$

Substituting these values in (1) of § 61, we have

$$x^2 + y^2 - 14x - 4y - 5 = 0$$

as the required equation.

13. Find the equation of a circle touching each axis at the distance −7 from the origin.

Here (a, b) is $(-7, -7)$ and $r = 7$. Substituting these values in [15], we obtain

$$(x + 7)^2 + (y + 7)^2 = 49,$$

or

$$x^2 + y^2 + 14x + 14y + 49 = 0,$$

as the required equation.

14. Find the equation of a circle touching both axes, and radius $= r$.

Here $r = r$, and (a, b) is (r, r), $(-r, r)$, $(-r, -r)$, or $(r, -r)$. Substituting these values successively in [15], we obtain

$$(x - r)^2 + (y - r)^2 = r^2, \text{ or } x^2 + y^2 - 2rx - 2ry + r^2 = 0;$$

$$(x + r)^2 + (y - r)^2 = r^2, \text{ or } x^2 + y^2 + 2rx - 2ry + r^2 = 0;$$

$$(x+r)^2+(y+r)^2=r^2, \text{ or } x^2+y^2+2rx+2ry+r^2=0;$$

and $$(x-r)^2+(y+r)^2=r^2, \text{ or } x^2+y^2-2rx+2ry+r^2=0,$$

as the required four equations.

15. Find the equation of a circle: centre (a, a), and cutting chord $= b$ from each axis.

Here the radius r, the abscissa or ordinate of centre a, and one-half the chord b form a right triangle of which r is the hypotenuse; hence we have

$$r^2=a^2+\tfrac{1}{4}b^2, \text{ and } (a, b) \text{ is } (a, a).$$

Substituting these values in [15], we have

$$(x-a)^2+(y-a)^2=a^2+\tfrac{1}{4}b^2,$$

or $$x^2+y^2-2ax-2ay+a^2-\tfrac{1}{4}b^2=0,$$

as the required equation.

16. Find the equation of a circle having the centre $(0, 0)$, and touching $y=2x+3$.

Writing $y=2x+3$ in the normal form, we have

$$-\frac{2x}{\sqrt{5}}+\frac{y}{\sqrt{5}}=\frac{3}{\sqrt{5}}.$$

Hence the distance of (1) from the origin is $\frac{3}{5}\sqrt{5}$, and therefore

$$r=\tfrac{3}{5}\sqrt{5}.$$

Substituting this value of r in [16], we have

$$x^2+y^2=\tfrac{9}{5}$$

as the required equation.

17. Find the equation of a circle having the centre $(1, -3)$, and touching $2x-y-4=0$.

By [12] the distance of $2x-y-4=0$ from $(1, -3)$ is

$$\frac{2+3-4}{\sqrt{5}}, \text{ or } \tfrac{1}{5}\sqrt{5}.$$

Hence the radius of the required circle is $\frac{1}{5}\sqrt{5}$, and its centre is $(1, -3)$. Substituting these values in [15], we obtain

$$(x-1)^2+(y+3)^2=\tfrac{1}{5},$$

or $$5(x^2+y^2)-10x+30y+49=0,$$

as the required equation.

18. Find the equation of a circle with its centre in the line

$$5x - 7y - 8 = 0,$$

and touching the lines $2x - y = 0$, $x - 2y - 6 = 0$.

The centre must be in the bisectors of the angles between the lines $2x - y = 0$ and $x - 2y - 6 = 0$. By [14] the equations of these bisectors are

$$\frac{2x-y}{\sqrt{5}} = \pm\frac{x-2y-6}{\sqrt{5}},$$

or $\quad x + y + 6 = 0, \qquad (1)$

and $\quad x - y = 2. \qquad (2)$

The centre is also on the line

$$5x - 7y - 8 = 0. \qquad (3)$$

The common point of (1) and (3) is $(-\frac{17}{6}, -\frac{19}{6})$; hence the centre of one circle is $(-\frac{17}{6}, -\frac{19}{6})$.

By [12] the distance of this centre from $2x - y = 0$ is $\dfrac{-\frac{34}{6}+\frac{19}{6}}{\sqrt{5}}$ or $-\frac{1}{2}\sqrt{5}$; hence the radius is $\frac{1}{2}\sqrt{5}$.

Making these substitutions in [15], we have

$$(x + \tfrac{17}{6})^2 + (y + \tfrac{19}{6})^2 = \tfrac{5}{4}. \qquad (4)$$

The common point of (2) and (3), or the centre of the second circle, is (3, 1). The distance of this centre from $2x - y = 0$ is $\dfrac{6-1}{\sqrt{5}}$, or $\sqrt{5}$; hence $r^2 = 5$, and the equation of the second circle is

$$(x - 3)^2 + (y - 1)^2 = 5.$$

19. Find the equation of a circle passing through the origin and the points common to the circles

$$x^2 + y^2 - 6x - 10y - 15 = 0,$$

$$x^2 + y^2 + 2x + 4y - 20 = 0.$$

By § 51 the required equation may be written in the form

$$x^2 + y^2 - 6x - 10y - 15 + k(x^2 + y^2 + 2x + 4y - 20) = 0. \qquad (1)$$

Since (1) passes through the origin, we have

$$-15 - 20k = 0, \text{ or } k = -\tfrac{3}{4}.$$

Substituting this value of k in (1), we obtain

$$x^2 + y^2 - 30x - 52y = 0$$

as the required equation.

20. Find the equation of a circle having its centre in the line

$$5x - 3y - 7 = 0,$$

and passing through the points common to the same circles as in No. 19.

Equation (1) of No. 19 may be written in the form

$$x^2 + y^2 - \frac{6 - 2k}{1 + k}x - \frac{10 - 4k}{1 + k}y = \frac{20k + 15}{1 + k}. \qquad (1)$$

From (1) we know that the centre of the required circle is

$$\left(\frac{3 - k}{1 + k}, \frac{5 - 2k}{1 + k}\right).$$

But this centre is in the line $5x - 3y - 7 = 0$. Hence we have

$$\frac{15 - 5k}{1 + k} - \frac{15 - 6k}{1 + k} = 7, \text{ or } k = -\tfrac{7}{6}.$$

Substituting this value of k in (1), or (1) of No. 19, we obtain, after reduction,

$$x^2 + y^2 + 50x + 88y - 50 = 0$$

as the required equation.

21. Find the equation of a circle touching the axis of x, and passing through the points common to the circles

$$x^2 + y^2 + 4x - 14y - 68 = 0,$$

$$x^2 + y^2 - 6x - 22y + 30 = 0.$$

By § 51, the required equation may be written in the form

$$x^2 + y^2 + 4x - 14y - 68 + k(x^2 + y^2 - 6x - 22y + 30) = 0. \qquad (1)$$

Putting (1) in the form of [15], we have

$$\left(x - \frac{3k - 2}{1 + k}\right)^2 + \left(y - \frac{11k + 7}{1 + k}\right)^2 = \frac{100k^2 + 180k + 121}{(1 + k)^2}$$

Hence $a = \dfrac{3k - 2}{1 + k}$, $b = \dfrac{11k + 7}{1 + k}$, $r^2 = \dfrac{100k^2 + 180k + 121}{(1 + k)^2}$.

But since the circle is tangent to the axis of x, we must have $b^2 = r^2$,

or $$\frac{(11k + 7)^2}{(1 + k)^2} = \frac{100k^2 + 180k + 121}{(1 + k)^2}. \qquad (2)$$

Solving (2) for k, we obtain $k = -\frac{4}{3}$ and $\frac{18}{7}$.

Substituting these values of k in (1), we obtain

$$x^2 + y^2 - 36x - 46y + 324 = 0,$$

and $$25x^2 + 25y^2 - 80x - 494y + 64 = 0,$$

as the required equations.

22. Find the centre and the radius of the circle which passes through (9, 6), (10, 5), (3, −2).

Substituting the co-ordinates of each point in (1) of § 61, we obtain

$$81 + 36 + 18D + 12E + F = 0, \qquad (1)$$

$$100 + 25 + 20D + 10E + F = 0, \qquad (2)$$

$$9 + 4 + 6D - 4E + F = 0. \qquad (3)$$

Solving these equations for D, E, and F, we obtain

$$D = -6,\ E = -2,\ F = 15.$$

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 - 12x - 4y + 15 = 0,$$

or

$$(x-6)^2 + (y-2)^2 = 25.$$

Hence (a, b) is (6, 2) and $r = 5$.

23. What is the distance from the centre of the circle passing through (2, 0), (8, 0), (5, 9) to the straight line joining (0, − 11) and (−16, 1)?

Substituting the co-ordinates of each of the points (2, 0), (8, 0), (5, 9) successively in (1) of § 61, we obtain

$$4 + 4D + F = 0, \qquad (1)$$

$$64 + 16D + F = 0, \qquad (2)$$

$$25 + 81 + 10D + 18E + F = 0. \qquad (3)$$

Solving these equations for D, E, and F, we obtain

$$D = -5,\ E = -4,\ F = 16.$$

Hence, by § 61, (a, b) is (5, 4).

The equation of the line through (0, − 11) and (− 16, 1) is, by [4],

$$\frac{y+11}{x} = \frac{1+11}{-16} = -\tfrac{3}{4},\ \text{or}\ 3x + 4y + 44 = 0. \qquad (4)$$

By [12] the distance from (5, 4) to (4) is

$$\frac{-15-16-44}{5},\ \text{or}\ -15.$$

24. What is the distance from the centre of the circle

$$x^2 + y^2 - 4x + 8y = 0$$

to the line $4x - 3y + 30 = 0$?

The centre of $x^2 + y^2 - 4x + 8y = 0$ is (2, − 4) by § 61.

By [12] the distance from (2, − 4) to $4x - 3y + 30 = 0$ is

$$\frac{-8-12-30}{5},\ \text{or}\ -10.$$

25. What portion of the line $y = 5x + 2$ is contained within the circle $x^2 + y^2 - 13x - 4y - 9 = 0$?

Writing the equation of the circle in the form of [15], we have

$$(x - \tfrac{13}{2})^2 + (y - 2)^2 = \tfrac{221}{4}. \qquad (1)$$

The centre of (1) is $(\frac{13}{2}, 2)$ and $r^2 = \frac{221}{4}$.

The distance from $(\frac{13}{2}, 2)$ to the line $y - 5x = 2$ is

$$\frac{2 - \frac{65}{2} - 2}{\sqrt{26}}, \text{ or } -\frac{65}{2\sqrt{26}}.$$

Now the radius, half the chord, and its distance from the centre form a right triangle of which r is the hypotenuse; hence

$$\text{Chord} = 2\sqrt{\frac{221}{4} - \left(\frac{65}{2\sqrt{26}}\right)^2} = \tfrac{1}{2}\sqrt{234}.$$

26. Through that point of the circle $x^2 + y^2 = 25$ for which the abscissa $= 4$ and the ordinate is negative, a straight line parallel to $y = 3x - 5$ is drawn. Find the length of the intercepted chord.

For $x = 4$ in $x^2 + y^2 = 25$, $y = \pm 3$; hence $(4, -3)$ is a point of the line; and its slope is 3, since it is parallel to $y = 3x - 5$. From [5] the equation of a line through $(4, -3)$ and having the slope 3 is

$$y + 3 = 3(x - 4), \text{ or } y = 3x - 15,$$

whose distance from the origin is $\dfrac{15}{\sqrt{10}}$; hence

$$\text{Chord} = 2\sqrt{25 - \left(\frac{15}{\sqrt{10}}\right)^2} = \sqrt{10}.$$

27. Through the point (x_1, y_1), within the circle $x^2 + y^2 = r^2$, a chord is drawn so as to be bisected at this point. What is its equation?

The equation of the line through $(0, 0)$ and (x_1, y_1) is

$$\frac{y}{x} = \frac{y_1}{x_1}. \qquad (1)$$

The chord is perpendicular to (1) at the point (x_1, y_1); hence its slope is $-\dfrac{x_1}{y_1}$, and its equation is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1), \text{ or } x_1x + y_1y = x_1^2 + y_1^2.$$

28. What relation must exist among the coefficients of the equation

$$A(x^2 + y^2) + Dx + Ey + C = 0,$$

(i.) in order that the circle may touch the axis of x?

(ii.) in order that the circle may touch the axis of y?

(iii.) in order that the circle may touch both axes?

This equation may be written in the form

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AC}{4A^2}. \qquad (1)$$

The centre (a, b) is $\left(-\frac{D}{2A}, -\frac{E}{2A}\right)$, and $r^2 = \frac{D^2 + E^2 - 4AC}{4A^2}$.

(i.) Circle (1) is tangent to the axis of x when $b = \pm r$, or $b^2 = r^2$; that is, when $E^2 = D^2 + E^2 - 4AC$, or $D^2 = 4AC$.

(ii.) Circle (1) is tangent to the axes of y, when $a^2 = r^2$; that is, when $D^2 = D^2 + E^2 - 4AC$, or $E^2 = 4AC$.

(iii.) Circle (1) is tangent to both axes when $a^2 = b^2 = r^2$; that is, when $D^2 = E^2 = D^2 + E^2 - 4AC$, or $D^2 = E^2 = 4AC$.

29. Under what condition will the straight line $y = mx + c$ touch the circle $x^2 + y^2 = 2rx$?

The equation $x^2 + y^2 = 2rx$ may be written in the form

$$(x - r)^2 + y^2 = r^2. \qquad (1)$$

The centre of (1) is $(r, 0)$, and its radius is r.

The distance of $y = mx + c$ from $(r, 0)$ is, by [12],

$$\frac{-mr - c}{\sqrt{1 + m^2}}.$$

The line $y = mx + c$ will be tangent to (1) if this distance equals $\pm r$; that is, if

$$\frac{-(mr + c)}{\sqrt{1 + m^2}} = \pm r, \text{ or } r^2 = 2rmc + c^2.$$

30. What must be the value of k in order that the line $3x + 4y = k$ may touch the circle $y^2 = 10x - x^2$?

The equation of the circle may be written in the form

$$(x - 5)^2 + y^2 = 25. \qquad (1)$$

The centre of (1) is $(5, 0)$ and $r = 5$.

The distance of $3x + 4y = k$ from $(5, 0)$ is, by [12], $\pm\frac{15 - k}{5}$.

If this distance equals 5; that is, if $\pm\dfrac{15-k}{5}=5$, or $k=40$ or -10, the line is a tangent to the circle.

31. Find the equation of the circle which passes through the origin and cuts equal lengths a from the lines $x=y$, $x+y=0$.

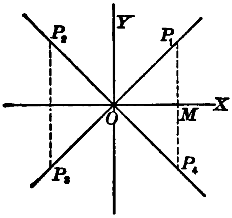

Fig. 53.

Let OX and OY be the axes and P_3P_1, P_4P_2 the lines $y=x$ and $y=-x$ respectively, which bisect the angles between the axes.

Let $\quad OP_1=OP_2=OP_3=OP_4=a;$

then $\quad OM=MP_1=\frac{1}{2}a\sqrt{2}.$

Hence P_1 is $(\frac{1}{2}a\sqrt{2},\ \frac{1}{2}a\sqrt{2})$,

P_2 is $(-\frac{1}{2}a\sqrt{2},\ \frac{1}{2}a\sqrt{2})$,

P_3 is $(-\frac{1}{2}a\sqrt{2},\ -\frac{1}{2}a\sqrt{2})$,

and $\quad P_4$ is $(\frac{1}{2}a\sqrt{2},\ -\frac{1}{2}a\sqrt{2})$.

The circle passing through O, P_1, and P_4 has its centre on OX; hence its equation is of the form

$$x^2+y^2=2rx. \qquad [17]$$

Since P_1 $(\frac{1}{2}a\sqrt{2},\ \frac{1}{2}a\sqrt{2})$ is on this circle, we must have

$$\tfrac{1}{2}a^2+\tfrac{1}{2}a^2=ra\sqrt{2}, \text{ or } r=\tfrac{1}{2}a\sqrt{2}.$$

Substituting this value of r in [17], we have

$$x^2+y^2=ax\sqrt{2}.$$

The circle passing through O, P_1, P_2 is $x^2+y^2=ay\sqrt{2}$.

The circle passing through O, P_2, P_3 is $x^2+y^2=-ax\sqrt{2}$.

The circle passing through O, P_3, P_4 is $x^2+y^2=-ay\sqrt{2}$.

32. Find the equations of the four circles whose common radius $=a\sqrt{2}$, and which cut chords from each axis equal to $2a$.

The radius, one-half the chord, and either co-ordinate of the centre form a right triangle of which the radius is the hypotenuse. Hence either co-ordinate of the centre equals $\sqrt{(a\sqrt{2})^2-a^2}$, or a numerically, and the four circles are

$$(x-a)^2+(y-a)^2=2a^2,$$
$$(x+a)^2+(y-a)^2=2a^2,$$
$$(x+a)^2+(y+a)^2=2a^2,$$
$$(x-a)^2+(y+a)^2=2a^2,$$

or

$$x^2+y^2\pm 2a(x\pm y)=0.$$

33. Find the equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 = r^2$, $(x - a)^2 + y^2 = r^2$.

Subtracting the first equation from the second, we find the equation of the common chord to be

$$x = \tfrac{1}{2}a. \tag{1}$$

The intersections of (1) and $x^2 + y^2 = r^2$ are $(\frac{1}{2}a, \frac{1}{2}\sqrt{4r^2 - a^2})$ and $(\frac{1}{2}a, -\frac{1}{2}\sqrt{4r^2 - a^2})$, which are the extremities of the chord, of which the middle point is $(\frac{1}{2}a, 0)$.

Hence the centre of the required circle is $(\frac{1}{2}a, 0)$, and its radius is $\frac{1}{2}\sqrt{4r^2 - a^2}$; therefore its equation is

$$(x - \tfrac{1}{2}a)^2 + y^2 = r^2 - \tfrac{1}{4}a^2,$$

or

$$2(x^2 - ax + y^2 - r^2) + a^2 = 0.$$

34. Find the equation of the straight line passing through (0, 0) and the centre of the circle $x^2 + y^2 = a(x + y)$.

From the form

$$x^2 - ax + y^2 - ay = 0,$$

we learn that the centre of the circle is $(\frac{1}{2}a, \frac{1}{2}a)$.

The line through (0, 0) and $(\frac{1}{2}a, \frac{1}{2}a)$ is, by [4],

$$\frac{y}{x} = \frac{\frac{1}{2}a}{\frac{1}{2}a} = 1, \text{ or } y = x.$$

35. Find the equation of the straight line passing through the centres of the circles $x^2 + y^2 = 25$ and $x^2 + y^2 + 6x - 8y = 0$.

The centres of the circles are (0, 0) and (−3, 4) respectively, and the equation of the line through these two points is

$$\frac{y}{x} = \frac{4}{-3}, \text{ or } 4x + 3y = 0.$$

36. Find the equation of the straight line passing through (0, 0) and touching the circle $x^2 + y^2 - 6x - 12y + 41 = 0$.

This equation may be written in the form

$$(x - 3)^2 + (y - 6)^2 = 4. \tag{1}$$

Hence (a, b) is (3, 6) and $r = 2$.

Since the line passes through the origin, its equation is of the form

$$y = mx. \tag{2}$$

The distance of (2) from (3, 6) is $\frac{3m-6}{\sqrt{1+m^2}}$, and (2) is a tangent to (1) if this distance equal 2; that is, if

$$\frac{3m-6}{\sqrt{1+m^2}}=2 \text{ or } (3m-6)^2=4+4m^2.$$

Whence $m=\frac{18}{5}\pm\frac{2}{5}\sqrt{41}$, and the required equation is

$$y=(\tfrac{18}{5}\pm\tfrac{2}{5}\sqrt{41})\,x, \text{ or } 5y=(18\pm 2\sqrt{41})\,x.$$

37. Find the equation of the straight line parallel to

$$x+\sqrt{3}\,(y-12)=0$$

and touching $x^2+y^2=100$.

Putting $x+\sqrt{3}\,(y-12)=0$ in the normal form, we have

$$\frac{x}{2}+\frac{\sqrt{3}}{2}y=6\sqrt{3}. \tag{1}$$

Now the required lines are parallel to (1), and at the distance 10 from the origin; hence their equations are

$$\frac{x}{2}+\frac{\sqrt{3}}{2}y=10 \text{ and } -\frac{x}{2}-\frac{\sqrt{3}}{2}y=10,$$

or

$$x+\sqrt{3}\,y\mp 20=0.$$

38. Find the equation of the straight line passing through the points common to the circles

$$x^2+y^2-2x-4y-20=0,$$
$$x^2+y^2-14x-16y+100=0.$$

Subtracting the second equation from the first, we obtain

$$12x+12y-120=0, \text{ or } x+y=10,$$

as the equation of the common chord of the circles.

39. Prove that the common chord of the circles in No. 38 is perpendicular to the straight line joining their centres.

The centre of the first circle is (1, 2), and of the second (7, 8), and the line through these points is

$$\frac{y-2}{x-1}=\frac{8-2}{7-1}=1, \text{ or } y=x+1. \tag{1}$$

The slope of (1) is + 1, and that of the common chord is − 1; hence, by § 45, Cor. 2, the lines are perpendicular.

40. Find the area of the triangle formed by radii of the circle $x^2 + y^2 = 169$ drawn to the points whose abscissas are -12 and $+7$ and ordinates positive, and the chord passing through the same two points.

For $x = -12$ in $x^2 + y^2 = 169$, $y = \pm 5$; hence one vertex is $(-12, 5)$.

For $x = 7$, $y = \pm 2\sqrt{30}$; hence a second vertex is $(7, 2\sqrt{30})$.

The third vertex is the origin.

By [13] the area of a triangle whose vertices are $(7, 2\sqrt{30})$, $(0, 0)$, $(-12, 5)$ is

$$\text{Area} = \tfrac{1}{2}[7(-5) - 12(2\sqrt{30})] = \tfrac{1}{2}(35 + 24\sqrt{30}).$$

41. Prove that an angle inscribed in a semicircle is a right angle.

Let $x^2 + y^2 = r^2$ be any circle; then the extremities of the horizontal diameter are $(-r, 0)$ and $(r, 0)$.

The equations of the chords through these points are

$$y = m(x + r), \tag{1}$$

$$y = m'(x - r). \tag{2}$$

Let the lines (1) and (2) intersect on the circle in (x_1, y_1); then we have

$$y_1 = m(x_1 + r), \tag{3}$$

$$y_1 = m'(x_1 - r), \tag{4}$$

$$y_1^2 = r^2 - x_1^2. \tag{5}$$

Multiplying (3) and (4) and dividing by (5) we obtain $mm' = -1$.

Hence lines (1) and (2) are perpendicular, if they intersect on the circle $x^2 + y^2 = r^2$

42. Prove that the radius of a circle drawn perpendicular to a chord bisects the chord.

Let P_1P_2 be any chord of the circle whose radius is r. Take as axes the diameters perpendicular and parallel to P_1P_2.

Then the equation of the circle is $x^2 + y^2 = r^2$, and

$$y = \pm\sqrt{r^2 - x^2},$$

from which it follows that the axis of x bisects P_1P_2 and every chord parallel to it.

43. Find the inclination to the axis of x of the line joining the centres of the circles $x^2 + 2x + y^2 = 0$, $x^2 + 2y + y^2 = 0$.

The centres of these circles are $(-1, 0)$ and $(0, -1)$.

By [4], the equation of the line through these two points is

$$\frac{y}{x+1} = \frac{-1}{1} = -1, \text{ or } y = -x-1$$

Here $\tan\gamma = m = -1$, $\therefore\ \gamma = 135^\circ$.

44. Determine the point from which tangents drawn to the circles

$$x^2 + y^2 - 2x - 6y + 6 = 0,$$

$$x^2 + y^2 - 22y - 20x + 52 = 0,$$

will each be equal to $4\sqrt{6}$.

The equations of the circles may be written

$$(x-1)^2 + (y-3)^2 = 4, \tag{1}$$

$$(x-10)^2 + (y-11)^2 = 169 \tag{2}$$

In (1), (a, b) is $(1, 3)$, and $r = 2$.

In (2), (a, b) is $(10, 11)$, and $r = 13$.

Let (x_1, y_1) be the required point; then the distance between (x_1, y_1) and $(1, 3)$ is the hypotenuse of a right triangle of which the other two sides are 2 and the length of the tangent. Hence we have

$$(x_1 - 1)^2 + (y_1 - 3)^2 = 2^2 + (4\sqrt{6})^2 = 100. \tag{3}$$

Also $$(x_1 - 10)^2 + (y_1 - 11)^2 = (13)^2 + (4\sqrt{6})^2 = 265. \tag{4}$$

Solving (3) and (4) for x_1 and y_1, we find as the required points $(7, -5)$ and $(-6\frac{7}{29}, 9\frac{26}{29})$.

45. Find the equations of the circles which touch the straight lines $6x + 7y + 9 = 0$ and $7x + 6y + 3 = 0$, and the latter line in the point $(3, -4)$.

Let (a, b) be the centres of the circles.

The distance of (a, b) from $6x + 7y + 9 = 0$ is

$$\frac{-6a - 7b - 9}{\sqrt{85}}.$$

The distance of (a, b) from $7x + 6y + 3 = 0$ is

$$\frac{-7a - 6b - 3}{\sqrt{85}}$$

Since the circles are tangent to both of these lines, these distances are numerically equal, and we have

$$\frac{-6a-7b-9}{\sqrt{85}} = \pm\frac{-7a-6b-3}{\sqrt{85}},$$

or $$a-b-6=0, \tag{1}$$

and $$13a+13b+12=0. \tag{2}$$

The line perpendicular to $7x+6y+3=0$ at $(3,-4)$ is, by § 46,

$$y+4=\tfrac{6}{7}(x-3), \text{ or } 7y-6x+46=0. \tag{3}$$

But the centre (a, b) is on (3); hence we have

$$7b-6a+46=0. \tag{4}$$

Solving (1) and (4) for a and b, we find one centre to be $(-4, -10)$.

Solving (2) and (4), we find the other centre to be $\left(\frac{514}{169}, -\frac{670}{169}\right)$.

The radius of the first is the distance from its centre $(-4, -10)$ to $(3, -4)$, or $\sqrt{85}$; hence its equation is

$$(x+4)^2+(y+10)^2=85.$$

The radius of the second is the distance from its centre

$$\left(\frac{514}{169}, -\frac{670}{169}\right) \text{ to } (3, -4), \text{ or } \frac{\sqrt{85}}{169};$$

hence its equation is

$$\left(x-\frac{514}{169}\right)^2+\left(y+\frac{670}{169}\right)^2=\frac{85}{169^2}.$$

46. Obtain and discuss the equation of the locus of the centre of a circle having the radius r and passing through the point (x_1, y_1).

Let (x, y) be the centre of the circle; then the distance from (x, y) to (x_1, y_1) equals r; that is,

$$(x-x_1)^2+(y-y_1)^2=r^2.$$

Hence the required locus is a circle whose centre is the point (x_1, y_1) and whose radius is r.

47. Obtain and discuss the equation of the locus of the centre of a circle having the radius r' and touching the circle

$$(x-a)^2+(y-b)^2=r^2.$$

The centre of this circle is (a, b), and its radius is r. Let (x, y) be the centre of the circle whose radius is r'.

If these circles are tangent externally, the distance between their centres equals the sum of their radii, and we have

$$(x-a)^2+(y-b)^2=(r+r')^2. \qquad (1)$$

If the circles are tangent internally, the distance between their centres equals the difference of their radii, and we have

$$(x-a)^2+(y-b)^2=(r-r')^2. \qquad (2)$$

Equations (1) and (2) are those required.

The locus of (1) is a circle whose centre is (a, b) and whose radius is $r+r'$, and that of (2) is a circle whose centre is (a, b) and whose radius is $\pm(r-r')$, the upper or lower sign being used according as $r >$ or $< r'$.

48. Obtain and discuss the equation of the locus of all points from which tangents drawn to the circle $(x-a)^2+(y-b)^2=r^2$ have a given length t.

Let (x, y) be the point from which tangents are drawn; then the distance from (x, y) to the centre (a, b) is the hypotenuse of a right triangle whose other sides are r and t; hence we have

$$(x-a)^2+(y-b)^2=r^2+t^2,$$

as the equation of the required locus, which is a circle whose centre is (a, b) and whose radius is $\sqrt{r^2+t^2}$.

49. Obtain and discuss the equation of the locus of the middle point of a chord drawn through a fixed point A of a given circle.

Let a diameter through A be the axis of x, and a tangent at A be the axis of y; then the centre of the given circle is $(r, 0)$, and its equation is

$$x^2+y^2=2rx. \qquad [17]$$

Let the equation of the chord be

$$y=mx. \qquad (1)$$

The line through the centre $(r, 0)$, and perpendicular to (1) is

$$y=-\frac{1}{m}(x-r), \text{ or } x+my=r. \qquad (2)$$

By No. 42, (2) bisects the chord intercepted on (1); hence the required locus is the locus of the intersection of (1) and (2). Combining these equations so as to eliminate m, we obtain

$$x^2+y^2=rx \qquad (3)$$

as the required equation.

The locus of (3) is a circle whose radius is $\frac{1}{2}r$ and whose centre is $(\frac{1}{2}r, 0)$.

50. Obtain and discuss the equation of the locus of the point M which divides the chord AC, drawn through the fixed point A of a given circle, in a given ratio

$$AM : MC = m : n.$$

Let a diameter through A be the axis of x, and a tangent at A be the axis of y; then the centre of the given circle is $(r, 0)$, and its equation is

$$x^2 + y^2 = 2rx. \qquad [17]$$

Let (x_2, y_2) be any point C on [17], and let (x, y) divide AC in the ratio $m : n$; then, by [3], we have

$$x = \frac{mx_2}{m+n}, \qquad y = \frac{my_2}{m+n}.$$

Whence $$x_2 = \frac{(m+n)x}{m}, \qquad y_2 = \frac{(m+n)y}{m}. \qquad (1)$$

Since (x_2, y_2) is on [17], we have

$$x_2^2 + y_2^2 = 2rx_2. \qquad (2)$$

Substituting in (2) the values of x_2 and y_2 in (1), we have after reduction

$$x^2 + y^2 = \frac{2mrx}{m+n} \qquad (3)$$

as the equation of the required locus.

The locus of (3) is a circle whose centre is $\left(\frac{mr}{m+n}, 0\right)$ and whose radius is $\frac{mr}{m+n}$.

For $m = n$, this gives the solution of No. 49.

51. Obtain and discuss the equation of the locus of a point whose distances from two fixed points, A, B, are in a constant ratio $m : n$.

Take A as the origin, AB as the axis of x, and let $AB = a$; then A is $(0, 0)$, and B is $(a, 0)$. Let (x, y) be the point whose locus is required. The distances of (x, y) from $(0, 0)$ and $(a, 0)$ are respectively

$$\sqrt{x^2 + y^2} \text{ and } \sqrt{(x-a)^2 + y^2}.$$

But these distances are in the ratio $m:n$; hence we have

$$\frac{x^2+y^2}{(x-a)^2+y^2}=\frac{m^2}{n^2},$$

or
$$(m^2-n^2)(x^2+y^2)-2am^2x+a^2m^2=0, \qquad (1)$$

as the equation of the locus sought.

The locus of (1) is a circle whose centre is $\left(\frac{am^2}{m^2-n^2},\ 0\right)$, and whose radius is $\frac{amn}{m^2-n^2}$.

52. Obtain and discuss the equation of the locus of a point, the sum of the squares of whose distances from two fixed points, A and B, is constant, and equal to k^2.

Take AB as the axes of x, and O, the middle point of AB, as the origin, and let $OA=OB=a$; then A is $(-a,\ 0)$, and B is $(a,\ 0)$. Let $(x,\ y)$ be the point whose locus is sought. The distances of $(x,\ y)$ from $(-a,\ 0)$ and $(a,\ 0)$ are respectively

$$\sqrt{(x+a)^2+y^2} \text{ and } \sqrt{(x-a)^2+y^2}.$$

But the sum of the squares of these distances is equal to k^2; hence we have

$$(x+a)^2+y^2+(x-a)^2+y^2=k^2,$$

or
$$x^2+y^2=\tfrac{1}{2}(k^2-2a^2), \qquad (1)$$

as the equation of the required locus.

The locus of (1) is a circle whose centre is $(0,\ 0)$, and whose radius is $\sqrt{\tfrac{1}{2}(k^2-2a^2)}$.

53. Obtain and discuss the equation of the locus of a point, the difference of the squares of whose distances from two fixed points, A, B, is constant and equal to k^2.

With the same axes and notation as in No. 52, we have

$$(x+a)^2+y^2-[(x-a)^2+y^2]=\pm k^2,$$

or
$$4ax=\pm k^2, \qquad (1)$$

as the equation of the required locus.

The locus of (1) is two lines parallel to the axis of y; one $k^2 \div 4a$ to the right, the other $k^2 \div 4a$ to the left, of that axis.

54. Obtain and discuss the equation of the locus of the middle point of a line of constant length d which moves so that its ends always touch two fixed perpendicular lines.

Let OX and OY be the fixed lines, and DB the line of length d in any position. Let (x, y) be P, the middle point of DB. Draw PM parallel to OY; then

$$MB^2 + MP^2 = PB^2. \qquad (1)$$

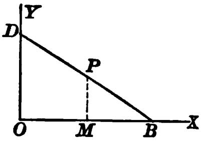

FIG. 54.

But $MB = OM = x$, $MP = y$, $PB = \frac{1}{2}d$.

Substituting in (1), we have

$$x^2 + y^2 = \tfrac{1}{4}d^2 \qquad (2)$$

as the equation of the required locus.

The locus of (2) is a circle whose centre is $(0, 0)$, and whose radius is $\frac{1}{2}d$.

55. Obtain and discuss the equation of the locus of the vertex of a triangle whose base is fixed and of constant length, and the angle at the vertex is also constant.

Let ACB be any one of the triangles constructed on AB as base, and having a vertical angle equal to ACB. Take O, the middle point of AB, as the origin, and OB as the axis of x.

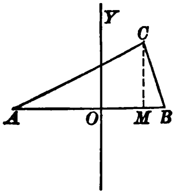

FIG. 55.

Let C be (x, y), $a = AO = OB$, $\theta = \angle ACB$, $\alpha = \angle BAC$, $\beta = \angle ABC$; then

$$\theta = 180^\circ - (\alpha + \beta),$$

and $$\tan\theta = -\tan(\alpha + \beta).$$

Whence $$\tan\theta = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \qquad (1)$$

But $$\tan\alpha = \frac{MC}{AM} = \frac{y}{a + x}, \quad \tan\beta = \frac{MC}{MB} = \frac{y}{a - x}.$$

Substituting these values in (1), we obtain

$$\tan\theta = \frac{\dfrac{y}{a+x} + \dfrac{y}{a-x}}{\dfrac{y^2}{a^2 - x^2} - 1},$$

or $$x^2 + y^2 - 2ay\cot\theta = a^2, \qquad (2)$$

as the equation of the required locus.

The locus of (2) is a circle whose centre is $(0, a\cot\theta)$, and whose radius is $a\csc\theta$.

56. One side, AB, of a triangle is constant in length and fixed in position; another side, AC, is constant in length, but revolves about the point A. Find the locus of the middle point of the third side, BC.

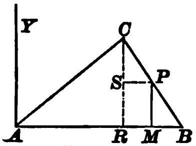

FIG. 56.

Let AC in the figure be one position of the revolving side, AB the fixed side, and P the middle point of the side BC. Let $AB = a$, $AC = b$; let (x, y) denote P referred to AB and AY.

Then $$AR^2 + RC^2 = AC^2 = b^2. \quad (1)$$

$$AR = AB - RB = a - 2\cdot MB = a - 2(a - x) = 2x - a.$$

$$RC = 2\cdot MP = 2y.$$

Substituting these values in (1), we obtain

$$(2x - a)^2 + (2y)^2 = b^2, \text{ or } (x - \tfrac{1}{2}a)^2 + y^2 = \tfrac{1}{4}b^2, \quad (2)$$

as the equation of the required locus.

The locus of (2) is a circle whose centre is $(\frac{1}{2}a, 0)$, and whose radius is $\frac{1}{2}b$.

57. Find the locus of the intersections of tangents at the extremities of a chord whose length is constant.

Let (x', y') be any point of intersection of tangents at the extremities of a chord whose length is l; then the equation of the chord of contact is

$$x'x + y'y = r^2. \quad (1)$$

The distance of (1) from the origin, or centre, is $\dfrac{r^2}{\sqrt{x'^2 + y'^2}}$. But this distance, half the chord l, and the radius form a right triangle of which the radius is the hypotenuse; hence we have

$$r^2 - \tfrac{1}{4}l^2 = \frac{r^4}{x'^2 + y'^2}, \text{ or } x'^2 + y'^2 = \frac{r^4}{r^2 - \frac{1}{4}l^2} = \frac{4r^4}{4r^2 - l^2}.$$

Since (x', y') is any point in the locus, and the square of its distance from the origin is $x'^2 + y'^2$ or $\dfrac{4r^4}{4r^2 - l^2}$, the equation of the locus is

$$x^2 + y^2 = \frac{4r^4}{4r^2 - l^2}.$$

Hence the locus is a circle.

58. A and B are any two fixed points, and P moves so that

$$PA = n \cdot PB;$$

find the locus of P.

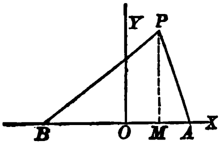

FIG. 57.

Let A and B be the fixed points, and take the line through them as the axis of x and the point mid-way between them as the origin. Let $OA = a$, and $P\,(x, y)$ be any position of the point P; then we have from the right triangles BMP and MPA

$$PA^2 = (a - x)^2 + y^2,$$
$$PB^2 = (a + x)^2 + y^2.$$

Substituting these values in the relation $PA^2 = n^2 \cdot PB^2$, we have

$$(a - x)^2 + y^2 = n^2 (a + x)^2 + n^2 y^2,$$

or

$$(1 - n^2)(x^2 + y^2 + a^2) - 2a(1 + n^2)x = 0,$$

of which the locus is evidently a circle.

Ex. 24. Page 97.

1. What is the equation of the diameter of the circle $x^2 + y^2 = 20$ which bisects chords parallel to the line $6x + 7y + 8 = 0$?

The equation of the diameter of the circle $x^2 + y^2 = r^2$, bisecting chords whose slope is m, is

$$y = -\frac{1}{m}x.$$

Here $m = -\frac{6}{7}$; hence

$$y = \tfrac{7}{6}x$$

is the required equation.

2. What is the equation of the diameter of the circle which bisects all chords whose inclination to the axis of x is 135°?

Here $m = -1$; hence the required equation is $y = x$.

3. Prove that the tangents at the extremities of a diameter are parallel.

If (x_1, y_1) be one extremity of a diameter of $x^2 + y^2 = r^2$, $(-x_1, -y_1)$ is the other extremity. By § 64 the slope of a tangent at (x_1, y_1) is $-\frac{x_1}{y_1}$; that of a tangent at $(-x_1, -y_1)$ is

$$-\frac{-x_1}{-y_1} = -\frac{x_1}{y_1}.$$

Their slopes being equal, the tangents are parallel.

Or, conceive the chord FE, Fig. 27 (Analytic Geometry), to move parallel to itself until the points F and E coincide at either extremity of the diameter. In either position it will be a tangent parallel to DC.

4. Write the equations of the chords of contact in the circle $x^2+y^2=r^2$ for tangents drawn from the following points:

$$(r, r),\ (2r, 3r),\ (a+b, a-b).$$

The equation of the chord of contact of tangents drawn from (h, k) to the circle $x^2+y^2=r^2$ is

$$hx+ky=r^2.$$

If (h, k) be (r, r), (1) becomes

$$rx+ry=r^2, \text{ or } x+y=r.$$

If (h, k) be $(2r, 3r)$, (1) becomes

$$2rx+3ry=r^2, \text{ or } 2x+3y=r.$$

If (h, k) be $(a+b, a-b)$, (1) becomes

$$(a+b)x+(a-b)y=r^2.$$

5. From the point (13, 2) tangents are drawn to the circle

$$x^2+y^2=49;$$

what is the equation of the chord of contact?

Here (h, k) is (13, 2) and $r^2=49$; hence the equation of the chord of contact is

$$13x+2y=49.$$

6. What line is represented by the equation $hx+ky=r^2$ when (h, k) is on the circle?

The tangent at (h, k).

7. Write the equations of the polars of the following points with respect to the circle $x^2+y^2=4$:

(i.) (2, 3). (ii.) (3, −1). (iii.) (1, −1)

The polar of the point (h, k) with regard to the circle $x^2+y^2=r^2$ is

$$hx+ky=r^2. \qquad (1)$$

(i.) If $r^2=4$, and (h, k) is (2, 3), (1) becomes

$$2x+3y=4.$$

(ii.) If $r^2 = 4$, and (h, k) is $(3, -1)$, (1) becomes

$$3x - y = 4.$$

(iii.) If $r^2 = 4$, and the pole is $(1, -1)$, the polar is

$$x - y = 4.$$

8. Find the poles of the following lines with respect to the circle $x^2 + y^2 = 35$:

(i.) $4x + 6y = 7$. (ii.) $3x - 2y = 5$. (iii.) $ax + by = 1$.

The pole of the line $hx + ky = r^2$ with respect to the circle $x^2 + y^2 = r^2$ is (h, k).

(i.) Since here $r^2 = 35$, to put $4x + 6y = 7$ in the form $hx + ky = r^2$ we multiply by 5, and obtain

$$20x + 30y = 35,$$

of which the pole is $(20, 30)$.

(ii.) $3x - 2y = 5$ in the form of $hx + ky = r^2$ is

$$21x - 14y = 35,$$

of which the pole is $(21, -14)$.

(iii.) $ax + by = 1$ put in the form of $hx + ky = r^2$ becomes

$$35ax + 35by = 35,$$

of which the pole is $(35a, 35b)$.

9. Find the pole of $3x + 4y = 7$ with respect to the circle

$$x^2 + y^2 = 14.$$

Here $r^2 = 14$; hence $3x + 4y = 7$ in the form of $hx + ky = r^2$ is

$$6x + 8y = 14,$$

and (h, k), on the pole, is $(6, 8)$.

10. Find the pole of $Ax + By + C = 0$ with respect to the circle

$$x^2 + y^2 = r^2.$$

Put in the form of $hx + ky = r^2$, $Ax + By + C = 0$ becomes

$$-\frac{Ar^2x}{C} - \frac{Br^2y}{C} = r^2,$$

and (h, k), or the pole with respect to $x^2 + y^2 = r^2$, is

$$\left(-\frac{Ar^2}{C}, -\frac{Br^2}{C}\right).$$

11. Find the co-ordinates of the points where the line $x = 4$ cuts the circle $x^2 + y^2 = 25$; also find the equations of the tangents at those points, and show that they intersect in the point $(\frac{25}{4}, 0)$.

For $x = 4$ in $x^2 + y^2 = 25$, $y = \pm 3$; hence in [18], $r^2 = 25$, and (x, y) is $(4, \pm 3)$. Substituting these values in [18], we obtain

$$4x \pm 3y = 25. \tag{1}$$

Regarding equations (1) as simultaneous, and solving for x and y, we find that their loci intersect in the point $(\frac{25}{4}, 0)$.

12. If the polars of two points P, Q meet in R, then R is the pole of the line PQ.

If the polars of P and Q each pass through R, the polar of R passes through each of the points P and Q (§ 74), and is therefore the line PQ.

13. If the polar of (h, k) with respect to the circle $x^2 + y^2 = r^2$ touch the circle $x^2 + y^2 = 2rx$, then $k^2 + 2rh = r^2$.

The polar of (h, k) with respect to $x^2 + y^2 = r^2$ is

$$hx + ky = r^2. \tag{1}$$

The tangent to $x^2 + y^2 = 2rx$ at (x_1, y_1) is

$$(x_1 - r)x + y_1y = rx_1. \tag{2}$$

If the polar (1) is a tangent to $x^2 + y^2 = 2rx$, (1) and (2) are the same line; hence we have

$$\frac{x_1 - r}{h} = \frac{y_1}{k}, \quad \frac{y_1}{k} = \frac{rx_1}{r^2} = \frac{x_1}{r}. \tag{3}$$

Solving equations (3), we obtain

$$y_1 = \frac{kr}{r - h}, \qquad x_1 = \frac{r^2}{r - h}. \tag{4}$$

Since (x_1, y_1) is on $x^2 + y^2 = 2rx$, we have

$$x_1^2 + y_1^2 = 2rx_1. \tag{5}$$

Substituting in (5) the values in (4), we obtain

$$k^2 + 2rh = r^2. \tag{6}$$

Hence the pole (h, k) is on the curve $y^2 + 2rx = r^2$, when the above conditions are fulfilled.

14. If the polar of (h, k) with respect to the circle $x^2 + y^2 = c^2$ touch the circle $4(x^2 + y^2) = c^2$; then the pole (h, k) will lie on the circle $x^2 + y^2 = 4c^2$.

The polar of (h, k) with respect to the circle $x^2 + y^2 = c^2$ is

$$hx + ky = c^2. \tag{1}$$

The tangent to $4(x^2 + y^2) = c^2$ at (x_1, y_1) is

$$4(x_1x + y_1y) = c^2. \tag{2}$$

If (1) and (2) are the same line, we have

$$4x_1 = h, \quad 4y_1 = k. \tag{3}$$

Since (x_1, y_1) is on $4(x^2 + y^2) = c^2$, we have

$$4(x_1^2 + y_1^2) = c^2. \tag{4}$$

From (3) and (4) we obtain $h^2 + k^2 = 4c^2$.
Hence (h, k) is on the circle $x^2 + y^2 = 4c^2$.

15. Find the polar of the centre of the circle $x^2 + y^2 = r^2$. Trace the changes in the position of the polar as the pole is supposed to move from the centre to an infinite distance.

In Fig. 32 of § 75, $OP \times OQ = OA^2$. As OP approaches zero, OQ increases without limit; hence the polar of the centre O is at infinity.

Since $OP \times OQ = OA^2$, as OP increases from zero, OQ diminishes from infinity. While OP increases from zero to OA, OQ decreases from infinity to OA. While OP increases from OA to infinity, OQ diminishes from OA to zero.

16. What is the square of the tangent drawn from the point (h, k) to the circle $x^2 + y^2 = r^2$?

The tangent t and the radius r are the sides of a right triangle of which the hypotenuse is the distance from (h, k) to $(0, 0)$; hence we have $h^2 + k^2 = r^2 + t^2$, or $t^2 = h^2 + k^2 - r^2$.

17. Find the length of the tangent drawn from (2, 5) to the circle

$$x^2 + y^2 - 2x - 3y - 1 = 0.$$

By § 76 the length of the tangent from (h, k) to the circle

$$x^2 + y^2 - 2x - 3y - 1 = 0$$

is

$$\sqrt{h^2 + k^2 - 2h - 3k - 1}.$$

Here (h, k) is (2, 5); hence the tangent equals

$$\sqrt{4 + 25 - 4 - 15 - 1}, \text{ or } 3.$$

18. Find the radical axis of the circles $(x+5)^2+(y+6)^2=9$, $(x-7)^2+(y-11)^2=16$.

From (3) of § 77 we have as the required equation,

$$(x+5)^2+(y+6)^2-9=(x-7)^2+(y-11)^2-16,$$

or $$12x+17y=51.$$

19. Find the radical axis of the circles $x^2+y^2+2x+3y-7=0$, $x^2+y^2-2x-y+1=0$.

From (3) of § 77 we have as the required equation,

$$x^2+y^2+2x+3y-7=x^2+y^2-2x-y+1,$$

or $$x+y-2=0.$$

20. Find the radical axis of the circles $x^2+y^2+bx+by-c=0$, $ax^2+ay^2+a^2x+b^2y=0$.

From (3) of § 77 we have as the required equation,

$$x^2+y^2+bx+by-c=x^2+y^2+ax+\frac{b^2}{a}y,$$

or $$(a^2-ab)x-(ab-b^2)y+ac=0.$$

21. Find the radical axis and length of the common chord of the circles $x^2+y^2+ax+by+c=0$, $x^2+y^2+bx+ay+c=0$.

From (3) of § 77 we have as the required equation,

$$x^2+y^2+ax+by+c=x^2+y^2+bx+ay+c,$$

or $$x-y=0. \qquad (1)$$

The centre of the first circle is $(-\frac{1}{2}a, -\frac{1}{2}b)$, and its radius is $\frac{1}{2}\sqrt{a^2+b^2-4c}$. By [12] the distance from $(-\frac{1}{2}a, -\frac{1}{2}b)$ to the radical axis, or common chord, (1) is

$$\frac{-\frac{1}{2}a+\frac{1}{2}b}{\sqrt{2}}, \text{ or } \frac{b-a}{2\sqrt{2}}.$$

Now this distance and half the chord are the sides of a right triangle of which the hypotenuse is the radius of the first circle; hence

$$\text{Chord}=2\sqrt{\frac{a^2+b^2-4c}{4}-\left(\frac{b-x}{2\sqrt{2}}\right)^2}=\sqrt{\tfrac{1}{2}(a+b)^2-4c}.$$

22. Find the radical centre of the three circles

$$x^2+y^2+4x+7=0, \qquad (1)$$

$$2x^2+2y^2+3x+5y+9=0, \qquad (2)$$

$$x^2+y^2+y=0. \qquad (3)$$

The radical axis of (1) and (2) is

$$x^2 + y^2 + 4x + 7 = x^2 + y^2 + \tfrac{3}{2}x + \tfrac{5}{2}y + \tfrac{9}{2},$$

or $$x - y + 1 = 0. \qquad (4)$$

The radical axis of (1) and (3) is

$$x^2 + y^2 + 4x + 7 = x^2 + y^2 + y,$$

or $$4x - y + 7 = 0. \qquad (5)$$

The common point of (4) and (5) is $(-2, -1)$, which is therefore the radical centre of the circles.

Ex. 25. Page 104.

1. Find the distances from the point P in Fig. 58 to the two axes.

$$NP = OM = OP \cos MOP = \rho \cos \theta,$$
$$MP = OP \sin MOP = \rho \sin \theta.$$

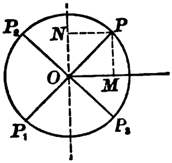

FIG. 58.

2. Prove that the equation of a straight line, referred to oblique axes in terms of its intercepts, is identical in form with [7].

If the axes were oblique in Fig. 19, § 38, AMP and AOB would be similar triangles, and we would have the same proportions as those used in § 40. Hence [7] holds for oblique axes.

3. If the straight line P_2OP_3 (Fig. 58) bisects the second and fourth quadrants, what are the polar co-ordinates of the points P_2 and P_3? Give more than one set of values in each case.

P_2 is $(a, \tfrac{3}{4}\pi)$, or $(-a, \tfrac{7}{4}\pi)$, or $(-a, -\tfrac{1}{4}\pi)$, or $(a, -\tfrac{5}{4}\pi)$;
P_3 is $(-a, \tfrac{3}{4}\pi)$, or $(a, \tfrac{7}{4}\pi)$, or $(a, -\tfrac{1}{4}\pi)$, or $(-a, -\tfrac{5}{4}\pi)$.

4. Construct the following points (on paper, take $a = 1$ in.):

$$\left(a, 0\right), \left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right), \left(-a, \frac{\pi}{2}\right), \left(-a, -\frac{\pi}{2}\right),$$
$$\left(2a, \frac{\pi}{6}\right), \left(2a, \pi\right), \left(a \cos\frac{\pi}{3}, \frac{\pi}{3}\right), \left(a, \frac{3\pi}{2}\right), \left(3a, \frac{2\pi}{3}\right),$$
$$\left(-3a, \frac{2\pi}{3}\right), \left(4a, \tan^{-1}\frac{4}{3}\right), \left(4a, \tan^{-1}\frac{3}{4}\right).$$

NOTE. The expression $\tan^{-1}\frac{4}{3}$ in higher Mathematics means "the angle whose tangent is $\frac{4}{3}$."

5. If ρ_1, ρ_2 denotes the two values of ρ in equation [25], p. 103 (Analytic Geometry), prove that $\rho_1\rho_2 = \rho'^2 - r^2$. What theorem of Elementary Geometry is expressed by this equation (i.) when the pole is outside the circle? (ii.) when the pole is inside the circle?

By Algebra we know that the product of the values of x in the quadratic $x^2 + ax + b = 0$ equals b; hence in [25]

$$\rho_1\rho_2 = \rho'^2 - r^2 = (\rho' + r)(\rho' - r). \quad (1)$$

(i.) In Fig. 59

$$\rho_1 = OP, \ \rho_2 = OQ,$$

$$\rho' + r = OM, \ \rho' - r = ON.$$

Substituting these values in (1), we have

$$OP \times OQ = OM \times ON,$$

or $$OP : OM :: ON : OQ.$$

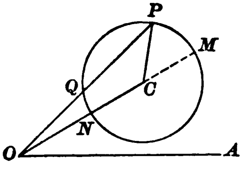

FIG. 59.

That is, if from a point without a circle two secants be drawn, the whole secants and the parts without the circle are reciprocally proportional.

(ii.) When the pole is within the circle, ρ_1 and ρ_2 are the two parts of any chord through the pole, and $\rho' + r$ and $\rho' - r$ are the parts of the diameter through the pole, ρ_2 and $\rho' - r$ being each negative. Hence if any chord and a diameter intersect each other in a circle, their segments are reciprocally proportional.

6. Through a fixed point P in a circle a chord PB is drawn, and then revolved about P; find the locus of its middle point.

Let P be the fixed point in the circle PBA, and PB the chord. Draw CQ perpendicular to PB; then Q is the middle point of PB, whose locus is sought. Let P be the pole, PC the polar axis, and Q the point (ρ, θ); then

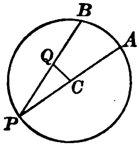

FIG. 60.

$$\cos\theta = \frac{PQ}{PC}, \text{ or } \rho = PC \times \cos\theta. \quad (1)$$

Comparing (1) with [24], we see that the locus of (1) is a circle whose diameter is PC.

If P were within or without the circle, we would obtain the same result, except that, when P is without the given circle, the locus consists only of that part of the circle whose diameter is PC, which lies within the given circle PBA.

7. If p denote the distance from the pole to a straight line, α the angle between p and the polar axis, prove that the polar equation of the line is $\rho \cos(\theta - \alpha) = p$.

Let O be the pole, OX the polar axis, OD the perpendicular on any line PD, and P any point (ρ, θ) on the line; then

$$OP = \frac{OD}{\cos DOP}, \text{ or } \rho = \frac{p}{\cos(\theta - \alpha)}$$

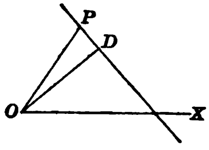

FIG. 61.

Ex. 26. Page 109.

1. What does the equation $y^2 - 4x + 4y + 8 = 0$ become when the origin is changed to the point $(1, -2)$?

To find the equation of a curve referred to new axes parallel to the old, the new origin being (h, k), we substitute $x + h$ for x and $y + k$ for y in the given equation of the curve. Here (h, k) is $(1, -2)$. Substituting $x + 1$ for x, and $y - 2$ for y in $y^2 - 4x + 4y + 8 = 0$, we obtain

$$(y-2)^2 - 4(x+1) + 4(y-2) + 8 = 0,$$

or

$$y^2 = 4x,$$

as the equation of the curve referred to the new axes.

2. Transform the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$ by changing the origin to the centre of the circle.

Here the new origin (h, k) is (a, b). Substituting $x + a$ for x, and $y + b$ for y in $(x-a)^2 + (y-b)^2 = r^2$, we obtain

$$(x + a - a)^2 + (y + b - b)^2 = r^2, \text{ or } x^2 + y^2 = r^2,$$

as the equation of the circle referred to its centre.

3. Transform the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$ by changing the origin to the left-hand end of the horizontal diameter.

Here the new origin (h, k) is $(a - r, b)$. Substituting $x + a - r$ for x, and $y + b$ for y in the given equation, we obtain

$$(x + a - r - a)^2 + (y + b - b)^2 = r^2, \text{ or } x^2 + y^2 = 2rx,$$

as the equation of the circle referred to the new axes.

4. Transform the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$ by changing the origin to the upper end of the vertical diameter.

Here the new origin (h, k) is $(a, b+r)$. Substituting $x+a$ for x, and $y+b+r$ for y in the given equation, we obtain

$$(x+a-a)^2+(y+b+r-b)^2=r^2, \text{ or } x^2+y^2+2ry=0,$$

as the equation of the circle referred to the new axes.

5. What does the equation $x^2+y^2=r^2$ become if the axes are turned through the angle a?

To find the equation of a curve referred to a new set of rectangular axes, the origin remaining the same, and the new axis of x making an angle θ with the old axis of x, we substitute $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y in the given equation.

Here $\theta = a$. Substituting $x\cos a - y\sin a$ for x, and $x\sin a + y\cos a$ for y in $x^2+y^2=r^2$, we obtain

$$(x\cos a - y\sin a)^2 + (x\sin a + y\cos a)^2 = r^2,$$

or

$$(\cos^2 a + \sin^2 a)(x^2+y^2) = x^2+y^2=r^2,$$

as the equation of the circle referred to the new axes.

6. What does the equation $x^2-y^2=a^2$ become if the axes are turned through -45°?

Here $\theta=-45^\circ$; hence $\sin\theta = -\frac{1}{2}\sqrt{2}$, and $\cos\theta=\frac{1}{2}\sqrt{2}$.

Substituting $\frac{1}{2}\sqrt{2}x+\frac{1}{2}\sqrt{2}y$ for x, and $-\frac{1}{2}\sqrt{2}x+\frac{1}{2}\sqrt{2}y$ for y in $x^2-y^2=a^2$, we obtain

$$[\tfrac{1}{2}\sqrt{2}(x+y)]^2-[\tfrac{1}{2}\sqrt{2}(y-x)]^2=a^2, \text{ or } 2xy=a^2,$$

as the equation of the curve referred to the new axes.

7. The equation of a curve referred to rectangular axes is

$$x-xy-y=0.$$

Transform it to new axes, whose origin is the point $(-1, 1)$, the new axis of y bisecting two of the angles formed by the old axes.

To change the reference of a curve from one set of rectangular axes to another, the new origin being (h, k) and the new axis of x making the angle θ with the old, we substitute $h+x\cos\theta - y\sin\theta$ for x, and $k+x\sin\theta+y\cos\theta$ for y in the given equation. Here (h, k) is $(-1, 1)$ and $\theta=45^\circ$; hence $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$. Substituting $-1+\frac{1}{2}\sqrt{2}(x-y)$ for x, and $1+\frac{1}{2}\sqrt{2}(x+y)$ for y in $x-xy-y=0$, we obtain

$$-1+\tfrac{1}{2}\sqrt{2}(x-y)-[-1+\tfrac{1}{2}\sqrt{2}(x-y)]$$
$$[1+\tfrac{1}{2}\sqrt{2}(x+y)]-1-\tfrac{1}{2}\sqrt{2}(x+y)=0,$$

or

$$x^2-y^2+2=0,$$

as the equation of the curve referred to the new axes.

8. Change the following equations to polar equations, taking the pole at the origin and the polar axis to coincide with the axis of x:

$$\text{(i.)}\ x^2 + y^2 = a^2. \qquad \text{(ii.)}\ x^2 - y^2 = a^2.$$

To obtain the polar equation of a curve from its rectangular equation, the pole being at the origin and the polar axis coinciding with the positive direction of the axis of x, we substitute $\rho \cos \theta$ for x, and $\rho \sin \theta$ for y.

(i.) Making these substitutions in $x^2 + y^2 = a^2$, we obtain

$$\rho^2 \cos^2 \theta + \rho^2 \sin^2 \theta = a^2 \text{ or } \rho = \pm a,$$

as the required polar equation of the circle.

(ii.) Making these substitutions in $x^2 - y^2 = a^2$, we obtain

$$\rho^2 \cos^2 \theta - \rho^2 \sin^2 \theta = a^2, \text{ or } \rho^2 \cos 2\theta = a^2,$$

as the required polar equation of the curve $x^2 - y^2 = a^2$.

9. Find the polar equation of the curve $x^2 = 4ay$, (i.) taking the pole at the origin; (ii.) taking the pole at the point $(a, 0)$.

(i.) Substituting $\rho \cos \theta$ for x, and $\rho \sin \theta$ for y in $x^2 = 4ay$, we obtain

$$\rho^2 \cos^2 \theta = 4a\rho \sin \theta,$$

or

$$\rho = 4a \tan \theta \sec \theta,$$

as the required polar equation.

(ii.) To obtain the polar equation of a curve from its rectangular equation, the pole being (h, k) and the polar axis being parallel to the axis of x, we substitute $h + \rho \cos \theta$ for x and $k + \rho \sin \theta$ for y. Here (h, k) is $(a, 0)$. Substituting $a + \rho \cos \theta$ for x and $\rho \sin \theta$ for y in $x^2 = 4ay$, we obtain

$$(a + \rho \cos \theta)^2 = 4a\rho \sin \theta.$$

10. Change the following equations to rectangular co-ordinates, the origin coinciding with the pole, and the polar axis with the axis of x:

$$\text{(i.)}\ \rho = a, \quad \text{(ii.)}\ \rho = a \cos \theta, \quad \text{(iii.)}\ \rho^2 \cos 2\theta = a^2.$$

To find the rectangular equation of a curve from its polar equation, the pole being the origin and the polar axis the axis of x, we substitute $x^2 + y^2$ for ρ^2, and $y \div x$ for $\tan \theta$.

(i.) Substituting $x^2 + y^2$ for ρ^2, we obtain

$$\sqrt{x^2 + y^2} = a, \text{ or } x^2 + y^2 = a^2$$

as the rectangular equation of the curve.

(ii.) Since $x = \rho \cos\theta$, $\cos\theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}$.

Substituting this value for $\cos\theta$, and $x^2 + y^2$ for ρ^2, we obtain

$$\sqrt{x^2 + y^2} = \frac{ax}{\sqrt{x^2 + y^2}}, \text{ or } x^2 + y^2 = ax,$$

as the rectangular equation of the curve.

(iii.) $\rho^2 \cos 2\theta = a^2$ may be written

$$\rho^2 (\cos^2\theta - \sin^2\theta) = a^2. \qquad (1)$$

Since $y = \rho \sin\theta$, $\sin\theta = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}$, and by (ii.) $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$.

Substituting in (1) these values for $\sin\theta$ and $\cos\theta$, and $x^2 + y^2$ for ρ^2, we obtain

$$(x^2 + y^2)\frac{x^2 - y^2}{x^2 + y^2} = a^2, \text{ or } x^2 - y^2 = a^2,$$

as the rectangular equation of the curve.

11. Transform $x + y + 2 = 0$ by changing the origin to the point $(-2, 0)$.

To find the equation of a curve referred to new axes parallel to the old, the new origin being (h, k), we substitute $x + h$ for x, and $y + k$ for y, in the given equation of the curve.

Here (h, k) is $(-2, 0)$. Substituting $x - 2$ for x, and y for y, in $x + y + 2 = 0$, we obtain

$$x + y = 0$$

as the required equation.

12. Transform $2x - 5y - 10 = 0$ by changing the origin to the point $(5, -2)$.

Here (h, k) is $(5, -2)$. Substituting $x + 5$ for x, and $y - 2$ for y, in $2x - 5y - 10 = 0$, we obtain

$$2x - 5y + 10 = 0$$

as the required equation.

13. Transform $3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0$ by changing the origin to the point $(\frac{7}{2}, -4)$.

Here (h, k) is $(\frac{7}{2}, -4)$. Substituting $x + \frac{7}{2}$ for x, and $y - 4$ for y, in $3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0$, we obtain

$$12x^2 + 4y^2 + 16xy + 1 = 0$$

as the required equation.

14. Transform $x^2 + y^2 - 2x - 4y = 20$ by changing the origin to the point (1, 2).

Here (h, k) is (1, 2). Substituting $x + 1$ for x, and $y + 2$ for y, in $x^2 + y^2 - 2x - 4y = 20$, we obtain

$$(x+1)^2 + (y+2)^2 - 2(x+1) - 4(y+2) = 0,$$

or $$x^2 + y^2 = 25,$$

as the required equation.

15. Transform $x^2 - 6xy + y^2 - 6x + 2y + 1 = 0$ by changing the origin to the point $(0, -1)$.

Here (h, k) is $(0, -1)$. Substituting x for x, and $y - 1$ for y, in $x^2 - 6xy + y^2 - 6x + 2y + 1 = 0$, we obtain

$$x^2 - 6x(y-1) + (y-1)^2 - 6x + 2(y-1) + 1 = 0,$$

or $$x^2 - 6xy + y^2 = 0,$$

as the required equation.

16. Transform the equation $x^2 - y^2 + 6 = 0$ by turning the axes through 45°.

To find the equation of a curve referred to new rectangular axes having the same origin as the old, the new axis of x making an angle θ with the old axis of x, we substitute $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y, in the given equation.

Here $\theta = 45$; hence $\sin\theta = \cos\theta = \frac{1}{2}\sqrt{2}$.

Substituting $\frac{1}{2}\sqrt{2}(x - y)$ for x, and $\frac{1}{2}\sqrt{2}(x + y)$ for y in

$x^2 - y^2 + 6 = 0$, we obtain

$$\tfrac{1}{2}(x-y)^2 - \tfrac{1}{2}(x+y)^2 + 6 = 0, \text{ or } xy = 3,$$

as the required equation.

17. Transform the equation $(x + y - 2a)^2 = 4xy$ by turning the axes through 45°.

Here $\theta = 45°$; hence $\cos a = \sin a = \frac{1}{2}\sqrt{2}$.

Substituting $\frac{1}{2}\sqrt{2}\,(x-y)$ for x, and $\frac{1}{2}\sqrt{2}\,(x+y)$ for y, in

$$(x+y-2a)^2=4xy,$$

we obtain

$$(\sqrt{2}x-2a)^2=2(x^2-y^2), \text{ or } y^2=2a(\sqrt{2}x-a),$$

as the required equation.

18. Transform the equation $9x^2-16y^2=144$ to oblique axes, such that the new axis of x makes with the old axis of x a negative angle whose tangent $=-\frac{3}{4}$; and the new axis of y makes with the old axis of x a positive angle whose tangent is $\frac{3}{4}$.

From the rectangular equation of a curve, to find its equation when referred to oblique axes having the same origin, α and β being the angles included between the old axis of x and the new axes of x and y respectively, we substitute $x\cos\alpha+y\cos\beta$ for x, and $x\sin\alpha+y\sin\beta$ for y, in the rectangular equation.

Here $\tan\alpha=-\frac{3}{4}$ and $\tan\beta=\frac{3}{4}$; hence $\cos\alpha=\frac{4}{5}$, $\sin\alpha=-\frac{3}{5}$, and $\cos\beta=\frac{4}{5}$, $\sin\beta=\frac{3}{5}$.

Substituting $\frac{4}{5}(x+y)$ for x, and $\frac{3}{5}(y-x)$ for y, in $9x^2-16y^2=144$, we obtain

$$4xy=25$$

as the required equation.

Ex. 27. Page 111.

1. Find the distance from the point $(-2b, b)$ to the origin, the axes making the angle 60°.

By § 79 the formula for the distance between the two points (x_1, y_1) and (x_2, y_2), when the angle between the axes is ω, is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+2(x_2-x_1)(y_2-y_1)\cos\omega}. \qquad (1)$$

Here (x_2, y_2) is $(-2b, b)$, (x_1, y_1) is $(0, 0)$, and $\omega=60°$.

Substituting these values in (1), we obtain

$$d=\sqrt{4b^2+b^2-2b^2}=b\sqrt{3}.$$

2. The axes making the angle ω, find the distance from the point $(1, -1)$ to the point $(-1, 1)$.

Here (x_1, y_1) is $(1, -1)$, and (x_2, y_2) is $(-1, 1)$. Substituting these values in (1) of No. 1, we obtain

$$d = 4\sqrt{\frac{1-\cos\omega}{2}} = 4\sin\tfrac{1}{2}\omega.$$

3. The axes making the angle ω, find the distance from the point $(0, 2)$ to the point $(3, 0)$.

Here (x_1, y_1) is $(0, 2)$ and (x_2, y_2) is $(3, 0)$. Substituting these values in (1) of No. 1, we obtain

$$d = \sqrt{13 - 12\cos\omega}.$$

4. Determine the distance between the points (a, θ) and (b, ϕ).

Let P and C (Fig. 40, in Analytic Geometry) be any two points (ρ_1, θ_1) and (ρ_2, θ_2); then from the triangle OCP we have

$$PC^2 = OP^2 + OC^2 - 2\,OP\cdot OC\cos COP.$$

Or, denoting the distance PC by d, we have

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2)}. \tag{1}$$

Here (ρ_1, θ_1) is (a, θ), and (ρ_2, θ_2) is (b, ϕ). Substituting these values in (1), we obtain

$$d = \sqrt{a^2 + b^2 - 2ab\cos(\theta - \phi)}.$$

5. Determine the distance between the points (a, θ) and $(a, -\theta)$.

Here (ρ_1, θ_1) is (a, θ), and (ρ_2, θ_2) is $(a, -\theta)$. Substituting these values in (1) of No. 4, we obtain

$$d = 2a\sqrt{\frac{1-\cos 2\theta}{2}} = 2a\sin\theta.$$

6. Determine the distance between the points (a, θ) and $(-a, -\theta)$.

Here (ρ_1, θ_1) is (a, θ), and (ρ_2, θ_2) is $(-a, -\theta)$. Substituting these values in (1) of No. 4, we obtain

$$d = 2a\sqrt{\frac{1+\cos 2\theta}{2}} = 2a\cos\theta.$$

7. Determine the distance between the points $(2a, 30^\circ)$ and $(a, 60^\circ)$.

Here (ρ_1, θ_1) is $(2a, 30^\circ)$, and (ρ_2, θ_2) is $(a, 60^\circ)$. Substituting these values in (1) of No. 4, we obtain

$$d = \sqrt{4a^2 + a^2 - 4a^2\cos(-30^\circ)} = a\sqrt{5 - 2\sqrt{3}}.$$

8. Show that the polar co-ordinates (ρ, θ), $(-\rho, \pi + \theta)$, $(-\rho, \theta - \pi)$ all represent the same point.

Suppose AOP, Fig. 38, § 82, equals θ, $OP = \rho$; then P is (ρ, θ), $AOP_1 = \pi + \theta$, and P is $(-\rho, \pi + \theta)$; also the negative angle $AOP_1 = \theta - \pi$, and P is $(-\rho, \theta - \pi)$.

9. Transform the equation $8x^2 + 8xy + 4y^2 + 12x + 8y + 1 = 0$ to the new origin $(-\frac{1}{2}, -\frac{1}{2})$.

Here (h, k) is $(-\frac{1}{2}, -\frac{1}{2})$. Substituting $x - \frac{1}{2}$ for x, and $y - \frac{1}{2}$ for y, in $8x^2 + 8xy + 4y^2 + 12x + 8y + 1 = 0$, we obtain

$$2x^2 + 2xy + y^2 = 1$$

as the required equation.

10. Transform the equation $6x^2 + 3y^2 - 24x + 6 = 0$ to the new origin $(2, 0)$.

Here (h, k) is $(2, 0)$. Substituting $x + 2$ for x, and y for y, in $6x^2 + 3y^2 - 24x + 6 = 0$, we obtain

$$2x^2 + y^2 = 6$$

as the required equation.

11. Transform the equation $\frac{x}{a} + \frac{y}{b} = 1$ by changing the origin to the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, and turning the axes through an angle ϕ, such that $\tan\phi = -\frac{b}{a}$.

To change the reference of a curve from one set of rectangular axes to another, the new origin being (h, k), and the new axis of x making the angle θ with the old, we substitute $h + x\cos\theta - y\sin\theta$ for x, and $k + x\sin\theta + y\cos\theta$ for y in the given equation. Here (h, k) is $(\frac{1}{2}a, \frac{1}{2}b)$, and $\tan\theta = \tan\phi = -\frac{b}{a}$; hence $\cos\theta = \frac{-a}{\sqrt{a^2 + b^2}}$, and $\sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$.

Substituting $\frac{1}{2}a - \frac{ax + by}{\sqrt{a^2 + b^2}}$ for x, and $\frac{1}{2}b + \frac{bx - ay}{\sqrt{a^2 + b^2}}$ for y, in $\frac{x}{a} + \frac{y}{b} = 1$ we obtain

$$\frac{1}{2} - \frac{x}{\sqrt{a^2 + b^2}} + \frac{by}{a\sqrt{a^2 + b^2}} + \frac{1}{2} + \frac{x}{\sqrt{a^2 + b^2}} - \frac{ay}{b\sqrt{a^2 + b^2}} = 1,$$

or $y = 0$, as the required equation.

12. Transform the equation $17x^2 - 16xy + 17y^2 = 225$ to axes which bisect the axes of the old system.

To effect this transformation, we substitute $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y, θ being the angle between the old and the new axes of x. Here $\theta = 45°$; hence $\sin\theta = \cos\theta = \frac{1}{2}\sqrt{2}$. Substituting $\frac{1}{2}\sqrt{2}(x - y)$ for x, and $\frac{1}{2}\sqrt{2}(x + y)$ for y, in $17x^2 - 16xy + 17y^2 = 225$, we obtain, as the required equation,

$$\tfrac{17}{2}(x-y)^2 - 8(x^2 - y^2) + \tfrac{17}{2}(x+y)^2 = 225,$$

or

$$9x^2 + 25y^2 = 225.$$

Transform the following rectangular equations to polar equations, the polar axis in each case coinciding with, or being parallel to, the axis of x, and the pole being at the point whose co-ordinates are given:

13. $x^2 + y^2 = 8ax$; the pole $(0, 0)$.

To obtain the polar equation of a curve from its rectangular equation, the pole being (h, k), and the polar axis being parallel to the axis of x, we substitute

$$h + \rho\cos\theta \text{ for } x, \text{ and } k + \rho\sin\theta \text{ for } y, \qquad (1)$$

in the rectangular equation.

Here (h, k) is $(0, 0)$. Substituting $\rho\cos\theta$ for x, and $\rho\sin\theta$ for y, in $x^2 + y^2 = 8ax$, we obtain as the required equation

$$\rho^2(\cos^2\theta + \sin^2\theta) = 8a\rho\cos\theta, \text{ or } \rho = 8a\cos\theta.$$

14. $x^2 + y^2 = 8ax$; the pole $(4a, 0)$.

Here (h, k) is $(4a, 0)$. Substituting $4a + \rho\cos\theta$ for x, and $\rho\sin\theta$ for y, in $x^2 + y^2 = 8ax$, we obtain

$$\rho^2 = 16a^2, \text{ or } \rho = \pm 4a,$$

as the required equation.

15. $y^2 - 6y - 5x + 9 = 0$; the pole $(\frac{5}{4}, 3)$.

Here (h, k) is $(\frac{5}{4}, 3)$. Substituting $\frac{5}{4} + \rho\cos\theta$ for x, and $3 + \rho\sin\theta$ for y, in $y^2 - 6y - 5x + 9 = 0$, we obtain

$$\rho^2\sin^2\theta - 5\rho\cos\theta = \tfrac{25}{4}.$$

16. $x^2 - y^2 - 4x - 6y - 54 = 0$; the pole $(2, -3)$.

Here (h, k) is $(2, -3)$. Substituting $2 + \rho\cos\theta$ for x, and $-3 + \rho\sin\theta$ for y, in

$$x^2 - y^2 - 4x - 6y - 54 = 0,$$

we obtain

$$\rho^2 = \frac{49}{\cos^2\theta - \sin^2\theta} = \frac{49}{\cos 2\theta} = 49 \sec 2\theta,$$

or $$\rho = \pm 7\sqrt{\sec 2\theta}.$$

17. $(x^2 + y^2)^2 = k^2(x^2 - y^2)$; the pole (0, 0).

Here (h, k) is (0, 0). Substituting $\rho \cos\theta$ for x, and $\rho \sin\theta$ for y, in $(x^2 + y^2)^2 = k^2(x^2 - y^2)$, we obtain

$$\rho^2 = k^2 \cos 2\theta, \text{ or } \rho = \pm k\sqrt{\cos 2\theta}.$$

Transform the following polar equations to rectangular equations, the origin being at the pole and the axis of x coinciding with the polar axis:

18. $\rho^2 \sin 2\theta = 2a^2$.

To obtain the rectangular equation of a curve from its polar equation, the pole being the origin and the polar axis the axis of x, we substitute $x^2 + y^2$ for ρ^2, and $\frac{y}{x}$ for $\tan\theta$.

The equation $\rho^2 \sin 2\theta = 2a^2$ may be written in the form

$$\rho^2(2 \sin\theta \cos\theta) = 2a^2. \qquad (1)$$

Since $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$, and $\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$. Substituting the values of ρ^2, $\sin\theta$, and $\cos\theta$ in (1), we obtain

$$xy = a^2$$

as the required equation.

19. $\rho = k \sin 2\theta$.

The equation $\rho = k \sin 2\theta$ may be written in the form

$$\rho = k(2 \sin\theta \cos\theta). \qquad (1)$$

Substituting in (1) the values of ρ^2, $\sin\theta$, and $\cos\theta$, as given in No. 18, we obtain as the required equation

$$(x^2 + y^2)^{\frac{3}{2}} = 2kxy.$$

20. $\rho(\sin 3\theta + \cos 3\theta) = 5k \sin\theta \cos\theta. \qquad (1)$

Since $\sin 3\theta = 3 \sin\theta \cos^2\theta - \sin^3\theta,$

and $\cos 3\theta = \cos^3\theta - 3 \sin^2\theta \cos\theta,$

equation (1) may be written in the form

$$\rho\,(3\sin\theta\cos^2\theta - \sin^3\theta + \cos^3\theta - 3\sin^2\theta\cos) = 5k\sin\theta\cos\theta. \quad (2)$$

Substituting in (2) the values of ρ, $\sin\theta$, and $\cos\theta$ given in No. 18, we obtain as the required equation

$$x^3 - y^3 + (3x - 3y - 5k)xy = 0.$$

21. Through what angle must a set of rectangular axes be turned in order that the new axis of x may pass through the point (5, 7)?

The slope of the new axis through (0, 0) and (5, 7) is $\frac{7}{5}$; hence the angle between the old and new axis of x is $\tan^{-1}\frac{7}{5}$.

22. The rectangular equation of a straight line is $Ax + By + C = 0$. Through what angle must the axes be turned in order

(i.) that the term containing x may disappear?
(ii.) that the term containing y may disappear?

The slope of the line $Ax + By + C = 0$ is $-\dfrac{A}{B}$.

(i.) That the term in x may disappear, the new axis of x must be parallel to the given line; hence $\theta = \tan^{-1}\left(-\dfrac{A}{B}\right)$.

(ii.) That the term in y may disappear, the new axis of x must be perpendicular to the given line; hence $\theta = \tan^{-1}\dfrac{B}{A}$.

23. Deduce the following formulas for changing from one set of oblique axes to another, the origin remaining the same:

$$x = \frac{x'\sin(\omega - \alpha)}{\sin\omega} + \frac{y'\sin(\omega - \beta)}{\sin\omega},$$

$$y = \frac{x'\sin\alpha}{\sin\omega} + \frac{y'\sin\beta}{\sin\omega}.$$

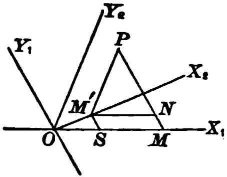

FIG. 62.

Let OX_1, OY_1 be the old axes, and OX_2, OY_2, the new. Let $X_1OX_2 = \alpha$, $X_1OY_2 = \beta$, and $X_1OY_1 = \omega$. Let P be any point, its co-ordinates referred to the old axes being $OM = x$, $MP = y$, its co-ordinates referred to the new axes being $OM' = x'$ and $M'P = y'$; then we have

$$x = OM = OS + SM$$

$$OM' \frac{\sin OM'S}{\sin OSM'} + M'P \frac{\sin M'PN}{\sin M'NP},$$

or $$x = x' \frac{\sin(\omega - \alpha)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega}. \qquad (1)$$

$$y = MP = SM' + NP$$

$$= OM' \frac{\sin SOM'}{\sin OSM'} + M'P \frac{\sin NM'P}{\sin M'NP},$$

or $$y = x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega}. \qquad (2)$$

24. From the formulas of No. 23 deduce those of § 88.

In § 88 the old axes are rectangular; hence $\omega = 90$, $\sin \omega = \sin 90^\circ = 1$, $\sin(\omega - \alpha) = \cos \alpha$, and $\sin(\omega - \beta) = \cos \beta$. Substituting these values in (1) and (2) of No. 23, we obtain

$$x = x' \cos \alpha + y' \cos \beta,$$

and $$y = x' \sin \alpha + y' \sin \beta.$$

Ex. 28. Page 117.

1. Show that the distance of any point of the parabola $y^2 = 4px$ from the focus is equal to $p + x$.

In Fig. 63 let P be any point (x, y) in the parabola $y^2 = 4px$, whose focus is F, and whose directrix is EC; then

$$FP = CP = DA + AM = p + x.$$

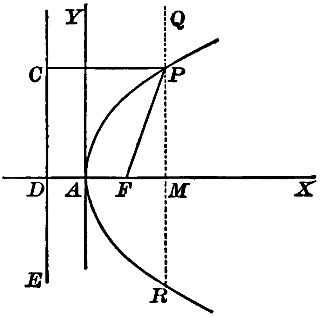

FIG. 63.

2. Find the equation of a parabola, taking as axes the axis of the curve and the directrix.

Let $P(x, y)$ be any point in the parabola RAP referred to the axis DX and the directrix DC. Let F be the focus; then in the right triangle FPM we have

$$FP^2 = FM^2 + MP^2. \qquad (1)$$

But $$FP = CP = DM = x,$$
$$FM = DM - DF = x - 2p,$$
$$MP = y.$$

Substituting these values in (1), we obtain
$$x^2 = (x - 2p)^2 + y^2,$$
or $$y^2 = 4p(x - p), \tag{2}$$
as the equation required.

3. Find the equation of a parabola, taking the axis of the curve as the axis of x and the focus as the origin.

Let $P(x, y)$ be any point in the parabola RAP referred to FX as the axis of x, and F as origin; then we have
$$FP^2 = FM^2 + MP^2. \tag{1}$$
$$FM = x, \; MP = y.$$
$$FP = CP = DM = DF + FM = 2p + x.$$

Substituting these values in (1), we obtain
$$(2p + x)^2 = x^2 + y^2,$$
or $$y^2 = 4p(x + p). \tag{2}$$

4. The distance from the focus of a parabola to the directrix $= 5$. Write its equation,

(i.) If the origin is taken at the vertex.
(ii.) If the origin is taken at the focus.
(iii.) If the axis and directrix are taken as axes.

Here $2p = 5$, or $p = \frac{5}{2}$.

(i.) Substituting this value of p in [26], we obtain
$$y^2 = 10x.$$

(ii.) Substituting this value of $2p$ in (2) of No. 3, we have
$$y^2 = 10x + 25.$$

(iii.) Substituting this value of $2p$ in (2) of No. 2, we have
$$y^2 = 10x - 25.$$

5. The distance from the focus to the vertex of a parabola is 4. Write its equations for the three cases enumerated in No. 4.

Here $p = 4$.

(i.) Substituting this value of p in [26], we obtain
$$y^2 = 16x.$$

(ii.) Substituting this value of p in (2) of No. 3, we obtain

$$y^2 = 16x + 64.$$

(iii.) Substituting this value of p in (2) of No. 2, we have

$$y^2 = 16x - 64.$$

6. For what point of the parabola $y^2 = 18x$ is the ordinate equal to three times the abscissa?

If in $y^2 = 18x$, we put $y = 3x$, we obtain

$$9x^2 = 18x, \text{ whence } x = 0, \text{ and } 2.$$

For $x = 0$ in $y^2 = 18x$, $y = 0$.
For $x = 2$ in $y^2 = 18x$, $y = \pm 6$.
Hence the points are (0, 0) and (2, 6).
The point (2, -6) is excluded since -6 is not three times 2.

7. Find the latus rectum for the following parabolas:

$$y^2 = 6x, \quad y^2 = 15x, \quad by^2 = ax.$$

Comparing each of these equations with $y^2 = 4px$, in which $4p$ is the latus rectum, we find that in

$$y^2 = 6x, \quad 4p = 6.$$
$$y^2 = 15x, \quad 4p = 15.$$
$$y^2 = \frac{a}{b}x, \quad 4p = \frac{a}{b}.$$

Find the points common to the following parabolas and straight lines:

8. $y^2 = 9x, \quad 3x - 7y + 30 = 0.$

Regarding these equations as simultaneous and solving for x and y, we find $x = 4$ and 25, $y = 6$ and 15; hence the common points of their loci are (4, 6) and (25, 15).

9. $y^2 = 3x, \quad x - 4y + 12 = 0.$

Regarding these equations as simultaneous and solving for x and y, we find $x = 12$, $y = 6$; hence the common point of their loci is (12, 6).

10. $y^2 = 4x, \quad x = 9, \quad x = 0, \quad x = -2.$

Regarding $y^2 = 4x$ and $x = 9$ as simultaneous, we find $x = 9$, $y = \pm 6$; hence their loci intersect in the points (9, 6) and (9, -6).

Regarding $y^2 = 4x$ and $x = 0$ as simultaneous, we find $x = 0$, $y = 0$; hence their loci have the origin as a common point.

Regarding $y^2 = 4x$ and $x = -2$ as simultaneous, we find $x = -2$, $y = \pm\sqrt{-8}$; hence their loci have no common point.

11. $y^2 = 4x$, $y = 6$, $y = -8$.

Regarding $y^2 = 4x$ and $y = 6$ as simultaneous, we find $y = 6$, $x = 9$; hence their loci have the point $(9, 6)$ in common.

Regarding $y^2 = 4x$ and $y = -8$ as simultaneous, we find $y = -8$, $x = 16$; hence their loci have the point $(16, -8)$ in common.

12. What must be the value of p in order that the parabola $y^2 = 4px$ may pass through the point $(9, -12)$?

If in $y^2 = 4px$ we put $x = 9$ and $y = -12$, we obtain

$$144 = 36p, \text{ or } p = 4.$$

Hence p must equal 4 in order that $y^2 = 4px$ may pass through the point $(9, -12)$.

13. For what point of the parabola $y^2 = 32x$ is the ordinate equal to 4 times the abscissa?

If in $y^2 = 32x$ we put $y = 4x$, we obtain $16x^2 = 32x$, whence $x = 0$ and 2.

For $x = 0$ in $y^2 = 32x$, $y = 0$.

For $x = 2$ in $y^2 = 32x$, $y = \pm 8$.

Hence the points are $(0, 0)$ and $(2, 8)$.

The point $(2, -8)$ is excluded, since -8 is not 4 times 2.

14. The equation of a parabola is $y^2 = 8x$. What is the equation of (i.) its axis, (ii.) its directrix, (iii.) its latus rectum, (iv.) a focal chord through the point whose abscissa = 8, (v.) a chord passing through the vertex and the negative end of the latus rectum?

Since the equation is of the form $y^2 = 4px$, the axes are the axis of the parabola and a tangent at its vertex. Hence,

(i.) Its axis is $y = 0$.

(ii.) Its directrix is $x = -2$, since $p = 2$.

(iii.) Its latus rectum is the line $x = 2$.

(iv.) For $x = 8$ in $y^2 = 8x$, $y = \pm 8$. The focus is $(2, 0)$.

The focal line to $(8, 8)$ is the line through $(2, 0)$ and $(8, 8)$, or

$$\frac{y}{x-2} = \frac{8}{6} = \frac{4}{3}, \text{ or } 3y = 4(x-2).$$

The focal line to $(8, -8)$ is the line from $(2, 0)$ to $(8, -8)$, or

$$\frac{y}{x-2} = -\frac{8}{6} = -\frac{4}{3}, \text{ or } 3y = -4(x-2).$$

Hence the focal lines are $4x \pm 3y - 8 = 0$.

(v.) The vertex is $(0, 0)$, and the negative end of the latus rectum is $(p, -2p)$ or $(2, -4)$. The line through these two points is

$$\frac{y}{x} = -\frac{4}{2} = -2, \text{ or } 2x + y = 0.$$

15. The equation of a parabola is $y^2 = 16x$. Find the equation of (i.) a chord through the points whose abscissas are 4 and 9, and ordinates positive; (ii.) the circle passing through the vertex and the ends of the latus rectum.

(i.) For $x = 4$ in $y^2 = 16x$, $y = \pm\ 8$,

For $x = 9$ in $y^2 = 16x$, $y = \pm 12$.

Hence the chord passes through the points $(4, 8)$ and $(9, 12)$, and its equation is

$$\frac{y-8}{x-4} = \frac{12-8}{9-4} = \frac{4}{5}, \text{ or } 4x - 5y + 24 = 0.$$

(ii.) Here $4p = 16$, or $p = 4$; hence the ends of the latus rectum are $(4, 8)$ and $(4, -8)$. The vertex is $(0, 0)$.

Substituting in (1) of § 61, the co-ordinates of each of the points $(0, 0)$, $(4, 8)$, $(4, -8)$, we obtain

$$F = 0. \tag{1}$$

$$16 + 64 + 8D + 16E + F = 0. \tag{2}$$

$$16 + 64 + 8D - 16E + F = 0. \tag{3}$$

Solving these equations for D, E, F, we obtain

$$D = -10,\ E = 0,\ F = 0.$$

Substituting these values in (1) of § 61, we obtain

$$x^2 + y^2 - 20x = 0$$

as the required equation.

16. If the distance of a point from the focus of the parabola $y^2 = 4px$ is equal to the latus rectum, what is the abscissa of the point?

By No. 1 the focal distance of any point (x, y) of the parabola $y^2 = 4px$ is $x + p$, and the latus rectum of $y^2 = 4px$ is $4p$.

Hence, if $x + p = 4p$, $x = 3p$.

17. In the parabola $y^2 = 4px$ an equilateral triangle is inscribed so that one vertex is at the origin. What is the length of one of its sides?

Since $y^2 = 4px$ is symmetrical with respect to the axis of x, and one vertex of the triangle is at the origin, the opposite side must be perpendicular to the axis of x; hence, if (x_1, y_1) be a second vertex $(x_1, -y_1)$ is the third, and the included side is $2y_1$. But another side is the distance from $(0, 0)$ to (x_1, y_1) or $\sqrt{x_1^2 + y_1^2}$.

Hence we have $2y_1 = \sqrt{x_1^2 + y_1^2}$, or $3y_1^2 = x_1^2$.

But since (x_1, y_1) is on $y^2 = 4px$, we have

$$y_1^2 = 4px_1.$$

Solving (1) and (2), we find $y_1 = 4p\sqrt{3}$; hence one side, or $2y_1$, equals $8p\sqrt{3}$.

18. A double ordinate of a parabola $= 8p$. Prove that straight lines drawn from its ends to the vertex are perpendicular to each other.

Since the double ordinate $= 8p$, the ordinates of its extremities are $\pm 4p$. For $y = \pm 4p$ in $y^2 = 4px$, $x = 4p$; hence the ends of the double ordinate are $(4p, 4p)$ and $(4p, -4p)$.

The slope of the line through $(0, 0)$ and $(4p, 4p)$ is $+1$; the slope of the line through $(0, 0)$ and $(4p, -4p)$ is -1; hence these two lines are perpendicular (§ 45, Cor. 2).

19. Explain how to construct a parabola, having given the directrix and the vertex.

Let A be the vertex and CE the directrix (Fig. 64). Through A draw an indefinite line perpendicular to CE, meeting CE in D. Lay off $AF = AD$; then F is the focus, and the problem is reduced to that of § 93.

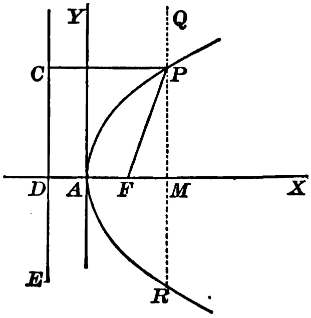

FIG. 64.

20. Explain how to construct a parabola, having given the focus and the vertex.

Let F be the focus and A the vertex (Fig. 64). Draw the indefinite line *DA*, lay off $AD = AF$, and through *D* draw *CE* perpendicular to *DF*; then *CE* is the directrix and *F* the focus, and the parabola may be drawn according to § 93.

21. Explain how to construct a parabola, having given the axis, vertex, and latus rectum.

Let *AX* (Fig. 64) be the axis, *A* the vertex, and $MQ =$ the latus rectum. From *A* lay off $AF = AD = \frac{1}{4} MQ$. Through *D* draw *CE* perpendicular to *AX*; then *CE* is the directrix, and *F* is the focus, and the curve may be drawn as in § 93.

22. Explain how to construct a parabola, having given the axis, vertex, and a point of the curve.

Let *AX* (Fig. 64) be the axis, *A* the vertex, and *P* any point of the curve. Draw *PM* perpendicular to *AX*. The third proportional to *AM* and *MP* will be the latus rectum (§ 97), and the problem is reduced to No. 21.

23. Explain how to construct a parabola, having given the axis, focus, and latus rectum.

Let *AX* (Fig. 64) be the axis, *F* the focus, and *MQ* the latus rectum. Take $FD = \frac{1}{2} MQ$, and through *D* draw *EC* perpendicular to *AX*; then *EC* is the directrix, and the curve may be drawn as in § 93.

24. Determine, as regards size and position, the relations of the following parabolas:

(i.) $y^2 = 4px$, (ii.) $y^2 = -4px$, (iii.) $x^2 = 4py$, (iv.) $x^2 = -4py$.

The latus rectum of each $= 4p$. The common vertex is at the origin. The axis of x is the axis of (i.) and (ii.); that of y is the axis of (iii.) and (iv.). Parabola (i.) lies wholly to the *right* of the origin, (ii.) wholly to the *left*, (iii.) wholly *above*, (iv.) wholly *below*. We may name them as follows:

(i.) is a right-handed *X*-parabola.
(ii.) is a left-handed *X*-parabola.
(iii.) is an upward *Y*-parabola.
(iv.) is a downward *Y*-parabola.

Ex. 29. Page 121.

1. The normal to a parabola at any point bisects the angle between the focal radius and the line drawn through the point parallel to the axis.

Let P be any point on the parabola AP whose focus is F. Let PT be a tangent at P, and draw PC parallel to AX. By Geometry,

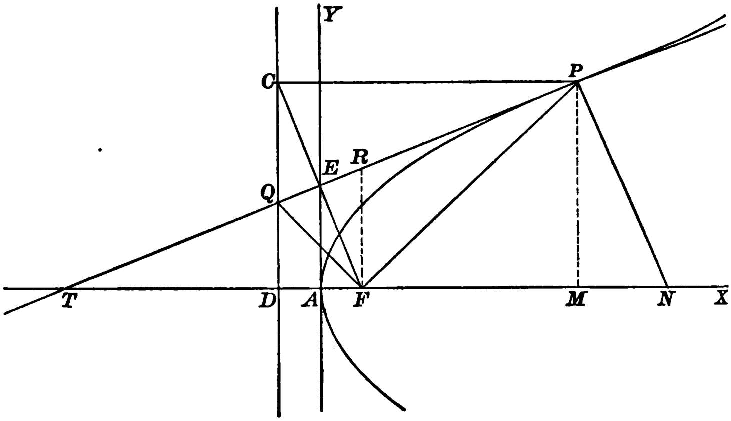

FIG. 65.

$\angle FTP = \angle TPC$, and by § 102, $\angle FTP = \angle TPF$; hence $\angle TPC = \angle TPF$; that is, the tangent PT bisects the angle FPC.

2. Explain how to draw a tangent and a normal to a given parabola at a given point.

Let P (Fig. 65) be any point on the parabola AP whose focus is F and whose axis is AX. Draw PC parallel to AX and draw PF. Bisect FPC with the line PT; then is PT a tangent at P by No. 1. Draw PN perpendicular to PT at P; then PN is the normal at P.

3. Prove that FC (Fig. 65) is perpendicular to PT.

Since $FP = CP$, triangle FPC is isosceles; hence PT, the bisector of the vertical angle FPC, is perpendicular to the base FC at its middle point.

4. Prove that the tangent $y = mx + \frac{p}{m}$ touches the parabola $y^2 = 4px$ at the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

Considering the two equations as simultaneous and solving for x and y, we find $x = \frac{p}{m^2}$, $y = \frac{2p}{m}$; hence the common point of their loci is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

Or, by trial we find that the co-ordinates of the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$ satisfy both equations; hence it is their common point.

5. Prove that the equation of a normal to the parabola $y^2 = 4px$ in terms of its slope is $y = mx - mp\,(2 + m^2)$.

The equation of the normal to $y^2 = 4px$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2p}(x - x_1). \qquad [28]$$

Denoting the slope of [28] by m, we have

$$m = -\frac{y_1}{2p}, \text{ or } y_1 = -2pm.$$

Hence $\quad 4p^2m^2 = y_1^2 = 4px_1$, or $x_1 = pm^2$.

Substituting these values in [28], we obtain

$$y + 2pm = m\,(x - pm^2),$$

or

$$y = mx - mp\,(2 + m^2),$$

as the equation of the normal in terms of its slope.

6. What are the equations of a tangent and a normal to the parabola $y^2 = 5x$, passing through the point whose abscissa is 20 and ordinate positive?

The equations of the tangent and normal to the parabola $y^2 = 4px$ at the point (x_1, y_1) are, respectively,

$$y_1y = 2p\,(x + x_1), \qquad [27]$$

and

$$y - y_1 = -\frac{y_1}{2p}(x - x_1). \qquad [28]$$

Here $\quad 4p = 5$.

For $x = 20$ in $y^2 = 5x$, $y = \pm 10$; hence the point of contact (x_1, y_1) is (20, 10). Substituting these values in [27] and [28], we obtain

$$10y = \tfrac{5}{2}(x + 20), \text{ or } x - 4y + 20 = 0;$$

and

$$y - 10 = -\frac{10}{\frac{5}{2}}(x - 20), \text{ or } 4x + y - 90 = 0,$$

as the equation of the tangent and normal respectively.

7. What are the equations of the tangents and the normals to the parabola $y^2 = 12x$, drawn through the ends of the latus rectum? Find the area of the figure which they enclose.

Here $4p = 12$, or $p = 3$, and the ends of the latus rectum $(p, 2p)$ and $(p, -2p)$ are $(3, 6)$ and $(3, -6)$; hence the point of contact (x_1, y_1) is $(3, 6)$ or $(3, -6)$. Substituting these values in [27], we obtain

$$6y = 6(x + 3), \text{ or } x - y + 3 = 0,$$

and $$-6y = 6(x + 3), \text{ or } x + y + 3 = 0.$$

Substituting these values in [28], we obtain

$$y - 6 = -\tfrac{6}{6}(x - 3), \text{ or } x + y - 9 = 0,$$

and $$y + 6 = \tfrac{6}{6}(x - 3), \text{ or } x - y - 9 = 0.$$

Since their slopes are negative reciprocals of each other, the tangents are perpendicular to each other, and the figure is a rectangle. Reducing these equations to the normal form, we see that the distance of each tangent from the origin is $\frac{3}{2}\sqrt{2}$, and the distance of each normal is $\frac{9}{2}\sqrt{2}$; hence the length of each side of this rectangle is $\frac{3}{2}\sqrt{2} + \frac{9}{2}\sqrt{2}$, or $6\sqrt{2}$; and the figure is a square whose area is $(6\sqrt{2})^2$, or 72.

8. Given the parabola $y^2 = 10x$. Through the point whose abscissa is 7 and ordinate positive a tangent and a normal are drawn. Find the lengths of the tangent, the normal, the subtangent, and the subnormal.

Here $4p = 10$. Since the subtangent is bisected at the vertex (§ 101), and $x_1 = 7$, subt $= 14$. Since subn $= 2p$, subn $= 5$.

For $x = 7$ in $y^2 = 10x$, $y = \pm\sqrt{70}$; hence $y_1 = \sqrt{70}$.

From Ex. 22, No. 3, we have

$$\text{Tan} = \sqrt{y_1^2 + \text{subt}^2} = \sqrt{70 + 196} = \sqrt{266};$$

$$\text{Nor} = \sqrt{y_1^2 + \text{subn}^2} = \sqrt{70 + 25} = \sqrt{95}.$$

9. A tangent to the parabola $y^3 = 20x$ makes with the axis of x an angle of 45°. Determine the point of contact.

The slope of the tangent to the parabola $y^2 = 4px$ is $2p \div y_1$. Here $4p = 20$, and the slope is $\tan 45°$, or 1. Hence $10 \div y_1 = 1$, or $y_1 = 10$.

For $y_1 = 10$ in $y_1^2 = 20x_1$, $x_1 = 5$; hence the point of contact is $(5, 10)$.

10. Show that the focus F (Fig. 65) is equidistant from the points P, T, N. What easy way of drawing a tangent and a normal is suggested by this theorem?

Since $\angle FTP = \angle FPT$, $FT = FP$.

Since PN and FC are both perpendicular to PT, they are parallel; hence the parallels CP and FN included between these parallels are equal. But $FP = CP$; hence $FP = FN$.

To draw a tangent and normal at P, draw FP, lay off

$$FT = FN = FP,$$

and draw PT and PN, which will be the tangent and normal at P.

11. If F is the focus of a parabola, and Q, R denote the points in which a tangent cuts the directrix and the latus rectum produced, prove that $FQ = FR$.

Since FR, AE, and DQ (Fig. 65) are parallel, and $AF = AD$, $ER = EQ$; hence $FR = FQ$, since they cut off equal distances from the foot of the perpendicular FE.

12. Prove that tangents drawn through the ends of the latus rectum are perpendicular to each other.

The ends of the latus rectum of $y^2 = 4px$ are $(p, 2p)$ and $(p, -2p)$, and the tangents at these points are

$$2py = 2p(x + p), \text{ or } y = x + p, \tag{1}$$

and

$$-2py = 2p(x + p), \text{ or } y = -x - p. \tag{2}$$

Lines (1) and (2) are perpendicular to each other by § 45, Cor. 2.

13. Find the distances of the vertex and the focus from the tangent

$$y = mx + \frac{p}{m}.$$

The distance from (x_1, y_1) to $y = mx + (p \div m)$ is, by [12],

$$\frac{-mx_1 + y_1 - (p \div m)}{\sqrt{1 + m^2}}. \tag{1}$$

If (x_1, y_1) be the vertex $(0, 0)$, the distance (1) becomes

$$\frac{p}{m\sqrt{1 + m^2}}.$$

If (x_1, y_1) be the focus $(p, 0)$, the distance (1) becomes

$$\frac{-m^2p - p}{m\sqrt{1 + m^2}} \text{ or } \frac{-p\sqrt{m^2 + 1}}{m}.$$

14. The points of contact of two tangents are (x_1, y_1) and (x_2, y_2). Find their point of intersection.

The equations of the two tangents are, by [27],

$$y_1y = 2p(x + x_1), \qquad (1)$$

$$y_2y = 2p(x + x_2). \qquad (2)$$

Regarding (1) and (2) as simultaneous, and solving,

$$x = \frac{y_1x_2 - y_2x_1}{y_2 - y_1} = \frac{x_2\sqrt{4px_1} - x_1\sqrt{4px_2}}{\sqrt{4px_2} - \sqrt{4px_1}} = \sqrt{x_1x_2},$$

and $$y = \frac{2px_1 - 2px_2}{y_1 - y_2} = \frac{y_1^2 - y_2^2}{2(y_1 - y_2)} = \tfrac{1}{2}(y_1 + y_2);$$

hence the tangents intersect in the point $(\sqrt{x_1x_2}, \tfrac{1}{2}[y_1 + y_2])$.

15. A tangent to the parabola $y^2 = 4px$ cuts equal intercepts on the axes. What is its equation? What is the point of contact? What is the value of the intercept?

The equation of the tangent to $y^2 = 4px$ at (x_1, y_1) is

$$y_1y = 2p(x + x_1). \qquad (1)$$

For $x = 0$ in (1), $y = \dfrac{2px_1}{y_1} =$ intercept on axis of y.

For $y = 0$ in (1), $x = -x_1 =$ intercept on axis of x.

Putting these intercepts equal to each other, we have

$$\frac{2px_1}{y_1} = -x_1, \text{ or } y_1 = -2p.$$

But $y_1^2 = 4px_1$; hence $4p^2 = 4px_1$, or $x_1 = p$.

Hence the point of contact is $(p, -2p)$.

Substituting these values of x_1 and y_1 in (1), we obtain

$$-2py = 2p(x + p), \text{ or } x + y + p = 0,$$

as the equation of the tangent.

Intercept $= -x_1 = -p$.

16. Through what point in the axis of x must tangents to the parabola $y^2 = 4px$ be drawn in order that they may form with the tangent, through the vertex, an equilateral triangle?

In Fig. 65 of No. 1, let $y_1y = 2p(x + x_1)$ be the equation of the tangent TP; then $AT = -x_1$, and $AE = \frac{2px_1}{y_1}$.

$$\therefore x_1^2 + \frac{4p^2x_1^2}{y_1^2} = TE^2.$$

But the triangle formed by the two tangents from T and the tangent AE is to be equilateral; hence we have

$$TE^2 = (2AE)^2 = \left(\frac{4px_1}{y_1}\right)^2.$$

Hence $$x_1^2 + \frac{4p^2x_1^2}{y_1^2} = \frac{16p^2x_1^2}{y_1^2}, \text{ or } y_1^2 = 12p^2.$$

$$\therefore 12p^2 = y_1^2 = 4px_1, \text{ or } x_1 = 3p.$$

But the intercept of the tangent on the axis of $x = -x_1 = -3p$; hence the tangents must be drawn from the point $(-3p, 0)$.

The points of contact are $(3p, \pm 2p\sqrt{3})$, and the tangents are

$$x \pm \sqrt{3}y + 3p = 0.$$

17. For what point of the parabola $y^2 = 4px$ is the normal equal to twice the subtangent?

By § 101, subn $= 2p$. $\therefore$ normal $= \sqrt{y_1^2 + 4p^2}$.

By § 101, subt $= 2x_1$.

If the normal equal twice the subtangent, we have

$$\sqrt{y_1^2 + 4p^2} = 4x_1, \text{ or } y_1^2 + 4p^2 = 16x_1^2. \quad (1)$$

Since (x_1, y_1) is on $y^2 = 4px$, we have

$$y_1^2 = 4px_1. \quad (2)$$

Solving (1) and (2), we find

$$x_1 = \tfrac{1}{8}p(1 \pm \sqrt{17}), \quad y_1 = \pm p\sqrt{\frac{1 \pm \sqrt{17}}{2}};$$

hence the points are

$$\left[\tfrac{1}{8}p(1 \pm \sqrt{17}), \quad \pm p\sqrt{\frac{1 + \sqrt{17}}{2}}\right].$$

The other two values of y are imaginary.

18. For what point of the parabola $y^2 = 4px$ is the normal equal to the difference between the subtangent and the subnormal?

Nor $= \sqrt{y_1^2 + 4p^2}$, subt $= 2x_1$, subn $= 2p$.

Hence, if nor $=$ subt $-$ subn, we have

$$\sqrt{y_1^2 + 4p^2} = 2x_1 - 2p. \qquad (1)$$

Since (x_1, y_1) is on the curve, we have

$$y_1^2 = 4px_1. \qquad (2)$$

Solving (1) and (2), we obtain $x_1 = 0,\ 3p,\ y_1 = 0,\ \pm 2p\sqrt{3}$; hence the required points are $(0, 0)$, $(3p,\ 2p\sqrt{3})$, and $(3p,\ -2p\sqrt{3})$.

19. Find the equation of a tangent to the parabola $y^2 = 5x$ parallel to the straight line $3x - 2y + 7 = 0$. Also find the point of contact.

By [27], the tangent to $y^2 = 5x$ at (x_1, y_1) is

$$yy_1 = \tfrac{5}{2}(x + x_1). \qquad (1)$$

The slope of (1) is $\dfrac{5}{2y_1}$, and the slope of $3x - 2y + 7 = 0$ is $\tfrac{3}{2}$. Since these lines are to be parallel, we have

$$\frac{5}{2y_1} = \frac{3}{2}, \text{ or } y_1 = \frac{5}{3}.$$

Since (x_1, y_1) is on $y^2 = 5x$, we have

$$5x_1 = y_1^2 = \tfrac{25}{9}, \text{ or } x_1 = \tfrac{5}{9}.$$

Hence the point of contact (x_1, y_1) is $(\tfrac{5}{9}, \tfrac{5}{3})$. Substituting these values in (1), we obtain

$$\tfrac{5}{3}y = \tfrac{5}{2}(x + \tfrac{5}{9}), \text{ or } 9x - 6y + 5 = 0.$$

20. Find the equation of the straight line which touches the parabola $y^2 = 12x$ and makes an angle of 45° with the line $y = 3x - 4$. Also find the point of contact.

By [27], the tangent to $y^2 = 12x$ at (x_1, y_1) is

$$y_1 y = 6(x + x_1). \qquad (1)$$

By § 47 the slope of lines making an angle of 45° with the line $y = 3x - 4$ is $\dfrac{3 \pm 1}{1 \mp 3}$; that is, the slope of the tangent must be -2 or $\tfrac{1}{2}$.

If $\dfrac{6}{y_1}$, or the slope of (1), equals -2, $y_1 = -3$.

But $\qquad 12x_1 = y_1^2 = 9$, or $x_1 = \tfrac{3}{4}$.

Hence one point of contact is $(\tfrac{3}{4}, -3)$.

If the slope of (1) equals $\frac{1}{2}$, $y_1 = 12$; hence

$$12x_1 = y_1^2 = 144, \text{ or } x_1 = 12,$$

and the second point of contact is (12, 12).

Substituting these values of x_1 and y_1 in (1), we obtain

$$4x + 2y + 3 = 0, \text{ and } x - 2y + 12 = 0,$$

as the equations of the tangents.

21. Find the equation of a straight line which touches the parabola $y^2 = 16x$ and passes through the point $(-4, 8)$.

By [27], the tangent to $y^2 = 16x$ at (x_1, y_1) is

$$y_1 y = 8(x + x_1). \qquad (1)$$

If (1) passes through $(-4, 8)$, we have

$$8y_1 = 8(-4 + x_1), \text{ or } y_1 = x_1 - 4. \qquad (2)$$

But $\qquad y_1^2 = 16x_1. \qquad (3)$

Solving (2) and (3), we obtain $x_1 = 12 \pm 8\sqrt{2}$, $y_1 = 8 \pm 8\sqrt{2}$.

Substituting these values in (1), we obtain

$$8(1 \pm \sqrt{2})y = 8(x + 12 \pm 8\sqrt{2}),$$

or $\qquad y = (-1 \pm \sqrt{2})x + 4(1 \pm \sqrt{2}),$

as the required tangent.

22. If a normal to a parabola meet the curve again in the point Q, find the length of PQ.

The normal to the parabola $y^2 = 4px$ at (x_1, y_1), or P, is

$$y - y_1 = -\frac{y_1}{2p}(x - x_1). \qquad (1)$$

Regarding (1) and $y^2 = 4px$ as simultaneous, we find that the second common point of their loci, or Q, is

$$\left[\frac{(2p + x_1)^2}{x_1}, \ -\frac{8p^2 + y_1^2}{y_1}\right].$$

Hence, by [1], we have

$$(\text{Chord } PQ)^2 = \left[x_1 - \frac{(2p + x_1)^2}{x_1}\right]^2 + \left(y_1 + \frac{8p^2 + y_1^2}{y_1}\right)^2,$$

$$= \frac{[-(4p^2 + 4px_1)]^2}{x_1^2} + \frac{(8px_1 + 8p^2)^2}{4px_1},$$

$$= \frac{16p(x_1 + p)^3}{x_1^2}.$$

$$\therefore \text{ Chord } PQ = \frac{4}{x_1}\sqrt{p(x_1 + p)^3}.$$

23. Prove by the secant method that the equation of a tangent to the parabola $y^2 = 4px - 4p^2$, at the point (x_1, y_1), is

$$y_1 y = 2p(x + x_1) - 4p^2.$$

Let (x_1, y_1) and (x_2, y_2) be any two points on the curve

$$y^2 = 4px - 4p^2.$$

The equation of the secant through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (2)$$

Since (x_1, y_1) and (x_2, y_2) are on (1), we have

$$y_1^2 = 4px_1 - 4p^2, \qquad (3)$$

$$y_2^2 = 4px_2 - 4p^2. \qquad (4)$$

Subtracting (3) from (4), we obtain

$$y_2^2 - y_1^2 = 4p(x_2 - x_1).$$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{4p}{y_2 + y_1}.$$

Substituting in (2), the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = \frac{4p}{y_2 + y_1}. \qquad (5)$$

If in (5) we put $y_2 = y_1$, we obtain as equation of the tangent

$$\frac{y - y_1}{x - x_1} = \frac{2p}{y_1}, \text{ or } yy_1 = 2p(x + x_1) - 4p^2.$$

24. Find the equations of the tangents and normals to the parabola $y^2 - 8x - 6y - 63 = 0$, drawn through the points whose common abscissa $= -1$.

Let (x_1, y_1) and (x_2, y_2) be any two points on the curve

$$y^2 - 8x - 6y - 63 = 0. \qquad (1)$$

Then $\quad y_1^2 - 6y_1 = 8x_1 + 63, \qquad (2)$

and $\quad y_2^2 - 6y_2 = 8x_2 + 63. \qquad (3)$

$$\therefore y_2^2 - y_1^2 - 6(y_2 - y_1) = 8(x_2 - x_1). \qquad (4)$$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{8}{y_2 + y_1 - 6}. \qquad (5)$$

Hence the equation of the secant through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{8}{y_2 + y_1 - 6}. \qquad (6)$$

Putting $y_2 = y_1$, we obtain as the equation of the tangent to (1) at (x_1, y_1),

$$\frac{y - y_1}{x - x_1} = \frac{4}{y_1 - 3}. \tag{6}$$

For $x = -1$ in (1), $y = 11$ and -5; hence the points of contact (x_1, y_1) are $(-1, 11)$ and $(-1, -5)$. Substituting these values of x_1 and y_1 in (6), we obtain

$$x - 2y + 23 = 0,$$

and $$x + 2y + 11 = 0,$$

as the required tangents.

25. What are the equations of tangents to the following parabolas:

(i.) $y^2 = -4px$? (ii.) $x^2 = 4py$? (iii.) $x^2 = -4py$?

(i.) $y^2 = -4px$ differs from $y^2 = 4px$ only in the sign before p; hence we may obtain the equation of the tangent to $y^2 = -4px$ by substituting $-p$ for p in $y_1y = 2p(x + x_1)$, and the required equation is

$$y_1y = -2p(x + x_1). \tag{1}$$

(ii.) $x^2 = 4py$ differs from $y^2 = 4px$ only in the exchange of places of x and y; hence we may obtain the equation of the tangent to $x^2 = 4py$, by substituting y for x, y_1 for x_1, x for y, and x_1 for y_1, in $y_1y = 2p(x + x_1)$; and the required equation is

$$x_1x = 2p(y + y_1). \tag{2}$$

(iii.) $x^2 = -4py$ differs from $x^2 = 4py$ only in the sign of p; hence by substituting $-p$ for p in (2), we obtain the equation of the tangent to $x^2 = -4py$, which is

$$x_1x = -2p(y + y_1).$$

Ex. 30. Page 123.

NOTE. If not otherwise specified, the axis of the parabola and the tangent at the vertex are to be assumed as axes of co-ordinates.

1. What is the equation of a parabola if the axis and directrix are taken as axes, and the focus is the point (12, 0)?

Since the directrix is the axis of y, and the focus is (12, 0), $2p = 12$. Substituting this value of $2p$ in $y^2 = 4p(x - p)$ (Eq. (2), No. 2, Ex. 28), we obtain

$$y^2 = 24(x - 6), \text{ or } y^2 = 24x - 144,$$

as the required equation.

2. What is the equation of a parabola if the axis and tangent at the vertex are the two axes, and (25, 20) is a point on the curve?

Since (25, 20) is on the curve $y^2 = 4px$, we must have $20^2 = 4 \times 25p$, or $p = 4$. Substituting this value of p in $y^2 = 4px$, we obtain as the required equation $y^2 = 16x$.

3. What is the equation of a parabola if the same axes are taken, and the focus is the point $(-4\frac{1}{4}, 0)$?

Since the focus is $(-4\frac{1}{4}, 0)$, the curve lies to the left of the axis of y, its equation is of the form $y^2 = -4px$, and $p = 4\frac{1}{4}$. Substituting this value of p in $y^2 = -4px$, we obtain as the required equation $y^2 = -17x$.

4. What is the equation of a parabola if the axis is parallel to the axis of x, the vertex is the point $(5, -3)$, and the latus rectum $= 5\frac{1}{2}$?

The equation of the curve referred to its axis and a tangent at its vertex is of the form $y^2 = 4px$, or $y^2 = -4px$.

Since the latus rectum $= 5\frac{1}{2}$, $4p = 5\frac{1}{2}$; hence the equation of the curve referred to its vertex is $y^2 = \frac{11}{2}x$, or $y^2 = -\frac{11}{2}x$.

Since the vertex is $(5, -3)$, the origin referred to the axes through the vertex is $(-5, 3)$.

Substituting $x - 5$ and $y + 3$ for x and y respectively in

$$y^2 = \tfrac{11}{2}x \text{ and } y^2 = -\tfrac{11}{2}x,$$

we obtain $$2y^2 - 11x + 12y + 73 = 0,$$

and $$2y^2 + 11x + 12y - 37 = 0,$$

as the required equations.

5. What is the equation of a parabola if the axis is the line $y = -7$, the abscissa of the vertex $= 3$, and one point is $(4, -5)$?

The vertex is evidently $(3, -7)$. Now, if (x_1, y_1) be the point $(4, -5)$ referred to the parallel axes through the vertex $(3, -7)$, we will have $4 = 3 + x_1$ and $-5 = -7 + y_1$, or (x_1, y_1) is $(1, 2)$. Since $(1, 2)$ lies to the right of the tangent at the vertex, the equation of the curve is of the form

$$y^2 = 4px. \qquad (1)$$

Since $(1, 2)$ is a point on (1), we have

$$4 = 4p, \text{ or } p = 1; \text{ hence } y^2 = 4x \qquad (2)$$

is the equation of the curve referred to its vertex. Since the vertex is $(3, -7)$, the origin referred to the axes through the vextex is $(-3, 7)$. Substituting $x - 3$ for x, and $y + 7$ for y, in (2), we obtain

$$(y + 7)^2 = 4(x - 3),$$

as the required equation.

6. What is the equation of a parabola if the curve passes through the points $(0, 0)$, $(3, 2)$, $(3, -2)$?

Since the curve passes through the origin, and since the points $(3, 2)$ and $(3, -2)$ are right symmetricals of each other with respect to the axis of x, and are to the right of the axes of y, the curve is a right-hand X-parabola, and its equation is of the form

$$y^2 = 4px. \tag{1}$$

Since $(3, 2)$ is on (1), we have $4 = 12p$, or $4p = \frac{4}{3}$.

Hence $y^2 = \frac{4}{3}x$, or $3y^2 = 4x$, is the required equation.

7. What is the equation of a parabola, if the curve passes through the points $(0, 0)$, $(3, 2)$, $(-3, 2)$?

Since the curve passes through the origin, and the points $(3, 2)$ and $(-3, 2)$ are right symmetricals of each other with respect to the axis of y, and above the axis of x, the curve is an upward Y-parabola, and its equation is of the form

$$x^2 = 4py. \tag{1}$$

Since $(3, 2)$ is on (1), we have

$$9 = 8p, \text{ or } 4p = \tfrac{9}{2}.$$

Hence $x^2 = \frac{9}{2}y$ is the equation required.

8. What is the latus rectum of the parabola $2y^2 = 3x$? What is the equation of its directrix, and of the focal chords passing through the points whose abscissa $= 6$?

Comparing $2y^2 = 3x$ with $y^2 = 4px$, we have latus rectum $= 4p = \frac{3}{2}$. Hence $p = \frac{3}{8}$, and the equation of the directrix is $x = -\frac{3}{8}$, or $8x + 3 = 0$.

For $x = 6$ in $2y^2 = 3x$, $y = \pm 3$. Hence the focal chords required are the line through $(6, 3)$ and $(\frac{3}{8}, 0)$, and the line through $(6, -3)$ and $(\frac{3}{8}, 0)$, or $8x \pm 15y - 3 = 0$.

9. Describe the change of form which the parabola $y^2 = 4px$ undergoes as we suppose p to diminish without limit.

As p diminishes indefinitely, DF (Fig. 46, § 93) diminishes indefinitely, and the branches of the curve traced by P approach nearer and nearer the axis DF, with which they coincide when $p = 0$. As p increases without limit, the branches approach nearer and nearer to the tangent at the vertex.

10. Find the intercepts of the parabola $y^2 + 4x - 6y - 16 = 0$. (1)

For $x = 0$ in (1), $y = 8$ and -2.

For $y = 0$ in (1), $x = 4$.

We can write (1) in the form

$$(y-3)^2 + 4(x - \tfrac{25}{4}) = 0. \tag{2}$$

Passing to parallel axes with origin $(\tfrac{25}{4}, 3)$, (2) becomes

$$y^2 = -4x.$$

Hence the locus of (1) is a left-hand X-parabola whose vertex is $(\tfrac{25}{4}, 3)$ and latus rectum 4.

11. One vertex of an equilateral triangle coincides with the focus, and the others lie in the parabola $y^2 = 4px$. Find the length of one side.

The side opposite the focus is evidently perpendicular to the axis of x; hence, if a second vertex is (x_1, y_1), the third is $(x_1, -y_1)$, and the side opposite the focus is $2y_1$. Another side is the distance from $(p, 0)$ to (x_1, y_1) or $\sqrt{(x_1 - p)^2 + y_1^2}$. Since the triangle is equilateral, we have

$$2y_1 = \sqrt{(x_1 - p)^2 + y_1^2}. \tag{1}$$

Since (x_1, y_1) is on $y^2 = 4px$, we have

$$y_1^2 = 4px_1. \tag{2}$$

Solving (1) and (2) as simultaneous, we find

$$x_1 = 7p \pm 4p\sqrt{3}$$

and $$y_1 = 2p(7 \pm 4\sqrt{3})^{\frac{1}{2}} = 2p\sqrt{4 \pm 2\sqrt{12} + 3},$$

or $$y_1 = 2p(2 \pm \sqrt{3}), \text{ and one side} = 2y_1 = 4p(2 \pm \sqrt{3}).$$

12. The latus rectum of a parabola $= 8$; find

(i.) Equation of a tangent through its positive end.
(ii.) Distance from the focus to this tangent.
(iii.) Equation of the normal at this point.

The equation is $y^2 = 8x$, since $4p = 8$.

(i.) The positive end of the latus rectum is $(p, 2p)$ or $(2, 4)$. Hence (x_1, y_1) is $(2, 4)$. Substituting these values in [27], we obtain

$$4y = 4(x + 2), \text{ or } y = x + 2. \qquad (1)$$

(ii.) By [12], the distance from the focus $(2, 0)$ to line (1) is

$$\frac{-2-2}{\sqrt{2}} \text{ or } -2\sqrt{2}.$$

(iii.) The equation of the perpendicular to (1) at $(2, 4)$ is

$$y - 4 = -1(x - 2), \text{ or } x + y - 6 = 0,$$

which is therefore the normal required.

13. What is the equation of the chord passing through the two points of the parabola $y^2 = 8x$ for which $x_1 = 2$, $y_1 > 0$, and $x_2 = 18$, $y_2 < 0$?

For $x = 2$ in $y^2 = 8x$, $y = \pm 4$; hence (x_1, y_1) is $(2, 4)$.
For $x = 18$ in $y^2 = 8x$, $y = \pm 12$; hence (x_2, y_2) is $(18, -12)$.
The line through $(2, 4)$ and $(18, -12)$ is

$$\frac{y-4}{x-2} = \frac{-16}{+16} = -1, \text{ or } x + y - 6 = 0.$$

14. Find the equation of the chord of the parabola $y^2 = 4px$ which is bisected at a given point (x_1, y_1).

Let (x_2, y_2) and (x_3, y_3) be the extremities of the chord; then by (2) of § 100 its slope is $\frac{4p}{y_2 + y_3}$, which is equal to $\frac{4p}{2y_1}$ or $\frac{2p}{y_1}$, since y_1 is the ordinate of the middle point of the chord. Now the equation of the line through (x_1, y_1) having the slope $\frac{2p}{y_1}$ is

$$y - y_1 = \frac{2p}{y_1}(x - x_1).$$

15. In what points does the line $x + y = 12$ meet the parabola

$$y^2 + 2x - 12y + 16 = 0? \qquad (1)$$

Regarding these equations as simultaneous, and solving for x and y, we find $x = 2, 8$, and $y = 10, 4$; hence the points are $(2, 10)$ and $(8, 4)$.

Writing (1) in the form $(y-6)^2+2(x-10)=0$, (2)
and passing to the parallel axes through (10, 6), (2) becomes

$$y^2=-2x.$$

Hence the locus of (1) is a left-handed X-parabola whose vertex is (10, 6) and whose latus rectum is 2.

16. In what points does the line $3y=2x+8$ meet the parabola

$$y^2-4x-8y+24=0? \qquad (1)$$

Regarding these equations as simultaneous, and solving for x and y, we find the points of intersection to be (2, 4) and (11, 10).

Writing (1) in the form $(y-4)^2=4(x-2)$, and passing to new parallel axes through (2, 4), we obtain $y^2=4x$; hence the locus of (1) is a right-hand X-parabola whose latus rectum is 4.

17. Find the equations of tangents from the origin to the parabola

$$(y-b)^2=4p(x-a). \qquad (1)$$

Since the tangents pass through the origin, their equations are of the form

$$y=mx. \qquad (2)$$

Regarding (1) and (2) as simultaneous, and solving for x, we obtain

$$x=\frac{mb+2p\pm\sqrt{-m^2(4pa+b^2)+(mb+2p)^2}}{m^2}. \qquad (3)$$

If (2) is a tangent, the two values of x in (3) must be equal; hence

$$(mb+2p)^2=m^2(4pa+b^2),$$

or

$$m=\frac{2p}{\pm\sqrt{4pa+b^2}-b}.$$

Substituting this value of m in (2), we have as the required equations

$$y=\frac{2px}{\pm\sqrt{4pa+b^2}-b}=\frac{b\pm\sqrt{b^2+4ap}}{2a}x.$$

If we pass to parallel axes whose origin is $(a,\ b)$, (1) becomes $y^2=4px$.

18. Describe the position of the parabola $y^2+2x+4=0$ with respect to the axes, and determine its latus rectum, vertex, focus and directrix.

The equation $y^2 + 2x + 4 = 0$, or the given equation may be written in the form $y^2 = -2(x + 2)$.

If we pass to new parallel axes, the new origin being $(-2, 0)$, $y^2 = -2(x + 2)$ becomes

$$y^2 = -2x. \tag{2}$$

Hence the locus of $y^2 = -2(x + 2)$ is a left-handed X-parabola, whose latus rectum is 2, whose vertex is $(-2, 0)$, whose focus is $(-2\frac{1}{2}, 0)$, whose axis is $y = 0$, and whose directrix is $x = -1\frac{1}{2}$.

19. What is the distance from the origin to a normal drawn through the end of the latus rectum of the parabola

$$y^2 = 4a(x - a)? \tag{1}$$

If we pass to new parallel axes, the new origin being $(a, 0)$, (1) becomes

$$y^2 = 4ax. \tag{2}$$

The normal to (2) through the point $(a, 2a)$ referred to the new axes is

$$y + x = 3a. \tag{3}$$

Now the distance from $(-a, 0)$ (the old origin referred to the new axes) to (3) is

$$\frac{-a - 3a}{\sqrt{2}}, \text{ or } -2a\sqrt{2}.$$

20. Find the equation of a parabola if the equation of a tangent is

$$4y = 3x - 12.$$

The equation of a tangent to $y^2 = 4px$ is $y_1 y = 2p(x + x_1)$,

or
$$y = \frac{2p}{y_1}(x + x_1). \tag{1}$$

Writing $4y = 3x - 12$ in the form of (1), we have

$$y = \tfrac{3}{4}(x - 4). \tag{2}$$

Comparing (1) and (2), we have

$$x_1 = -4, \ \frac{2p}{y_1} = \frac{3}{4}.$$

We have, also,

$$y_1^2 = 4px_1.$$

Eliminating x_1 and y_1 between these three equations, we obtain $4p = -9$; hence the required equation is $y^2 = -9x$.

21. Find the equation of a parabola if a focal radius $= 10$, and its equation is $3y = 4x - 8$.

For $y = 0$ in $3y = 4x - 8$, $x = 2$; hence the focus is $(2, 0)$, or $p = 2$, and the required equation is $y^2 = 8x$.

22. Find the equation of a parabola if for a point of the curve the focal radius $= r$, the length of the tangent $= t$.

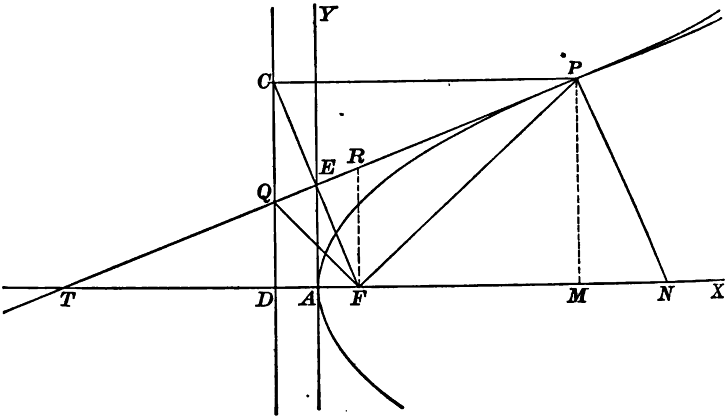

FIG. 66.

Let PT be a tangent at $P(x_1, y_1)$, $FP = r$, $TP = t$;

then $\qquad FP^2 = MP^2 + FM^2$,

or $\qquad r^2 = y_1^2 + (x_1 - p)^2$; $\qquad$ (1)

$\qquad TP^2 = TM^2 + MP^2$,

or $\qquad t^2 = (2x_1)^2 + y_1^2$; $\qquad$ (2)

and $\qquad y_1^2 = 4px_1$. $\qquad$ (3)

From (1) and (3),

$$r^2 = 4px_1 + (x_1 - p)^2 = (x_1 + p)^2, \text{ or } x_1 = r - p. \qquad (4)$$

From (2), (3), and (4),

$$t^2 = 4x_1^2 + 4px_1 = 4(r - p)^2 + 4p(r - p),$$

or $$4p = \frac{4r^2 - t^2}{r}.$$

Hence the required equation is

$$y^2 = \frac{4r^2 - t^2}{r} x.$$

23. Find the equation of a parabola if for a point of the curve the focal radius $= r$, the length of the normal $= n$.

If in Fig. 66, $NP = n$, $FP = r$, $y_1 = MP$, and $x_1 = AM$, or

$$FM = x_1 - p;$$

then we have

$$n^2 = y_1^2 + 4p^2, \tag{1}$$

$$r^2 = y_1^2 + (x_1 - p)^2, \tag{2}$$

and

$$y_1^2 = 4px_1. \tag{3}$$

From (2) and (3),

$$r^2 = (x_1 + p)^2, \text{ or } x_1 = r - p. \tag{4}$$

From (1), (3), and (4), we have

$$n^2 = 4p^2 + y_1^2 = 4p^2 + 4px_1 = 4p^2 + 4p(r - p).$$

$$\therefore n^2 = 4pr, \text{ or } 4p = n^2 \div r,$$

and the required equation is

$$y^2 = \frac{n^2}{r} x.$$

24. What is the equation of a parabola if for a point of the curve the length of the tangent $= t$, the length of the normal $= n$?

From Fig. 66, we have

$$t^2 = (2x_1)^2 + y_1^2, \tag{1}$$

$$n^2 = (2p)^2 + y_1^2. \tag{2}$$

Also,

$$y_1^2 = 4px_1. \tag{3}$$

From (2), $y_1^2 = n^2 - 4p^2$.

From (1) and (2),

$$t^2 - n^2 = 4x_1^2 - 4p^2, \text{ or } 2x_1 = \pm\sqrt{t^2 + 4p^2 - n^2}.$$

Substituting these values of y_1^2 and $2x_1$ in (3), we have

$$n^2 - 4p^2 = \pm 2p\sqrt{t^2 + 4p^2 - n^2}, \text{ or } p = \frac{n^2}{2\sqrt{t^2 + n^2}}.$$

Since p is a positive quantity, only the positive value of p is taken.

Substituting in [26], we have as the required equation

$$y^2 = \frac{2n^2}{\sqrt{t^2 + n^2}} x.$$

25. What is the equation of a parabola if for a point of the curve the focal radius $= r$, the subtangent $= s$?

Here subt $= 2x_1 = s$, or $x_1 = \frac{1}{2}s$.

$$y_1^2 = 4px_1 = 2ps.$$

Also, in Fig. 66,

$$FP^2 = FM^2 + MP^2, \text{ or } r^2 = (x_1 - p)^2 + y_1^2.$$

$$\therefore r^2 = (\tfrac{1}{2}s - p)^2 + 2ps = (\tfrac{1}{2}s + p)^2.$$

$$\therefore 4p = 4r - 2s = 2(2r - s).$$

Whence the required equation is

$$y^2 = 2(2r - s)x.$$

26. Two parabolas have the same vertex, and the same latus rectum $4p$, but their axes are perpendicular to each other. What is the length of their common chord?

Let one of the parabolas be $y^2 = 4px$; then we may take as the other $x^2 = 4py$. Regarding these equations as simultaneous, and solving, we find that their common points are (0, 0) and $(4p, 4p)$, the distance between which is $\sqrt{16p^2 + 16p^2}$, or $4p\sqrt{2}$.

27. Through the three points of the parabola $y^2 = 12x$, whose ordinates are 2, 3, 6, tangents are drawn. Show that the circle circumscribed about the triangle formed by the tangents passes through the focus.

For $y = 2, 3, 6$ in $y^2 = 12x$, $x = \frac{1}{3}, \frac{3}{4}, 3$; hence the points of contact are $(\frac{1}{3}, 2)$, $(\frac{3}{4}, 3)$, $(3, 6)$, and the tangents are

$$2y = 6(x + \tfrac{1}{3}), \text{ or } y = 3x + 1; \tag{1}$$

$$3y = 6(x + \tfrac{3}{4}), \text{ or } y = 2x + \tfrac{3}{2}; \tag{2}$$

and

$$6y = 6(x + 3), \text{ or } y = x + 3. \tag{3}$$

Regarding (1) and (2) as simultaneous, we find one vertex to be $(\frac{1}{2}, \frac{5}{2})$; regarding (2) and (3) as simultaneous, we find a second vertex to be $(\frac{3}{2}, \frac{9}{2})$; and regarding (3) and (1) as simultaneous, we find the third vertex to be (1, 4).

Substituting the co-ordinates of each vertex in

$$x^2 + y^2 + 2Dx + 2Ey + F = 0, \tag{4}$$

we obtain $\frac{1}{4} + \frac{25}{4} + D + 5E + F = 0,$ (5)

$$\tfrac{9}{4} + \tfrac{81}{4} + 3D + 9E + F = 0, \tag{6}$$

and $1 + 16 + 2D + 8E + F = 0.$ (7)

Solving (5), (6), and (7) for D, E, and F, we obtain

$$D = -3,\ E = -\tfrac{5}{2},\ F = 9.$$

Substituting these values in (4), we obtain

$$x^2 + y^2 - 6x - 5y + 9 = 0, \tag{8}$$

which is satisfied for $x = 3$ and $y = 0$; hence circle (8) passes through the focus (3, 0) of $y^2 = 12x$.

28. A tangent to the parabola $y^2 = 4px$ makes the angle 30° with the axis of x. At what point does it cut the axis?

The equation of a tangent to $y^2 = 4px$ is

$$y = mx + \frac{p}{m}. \tag{1}$$

Here $m = \tan 30^\circ = \frac{1}{3}\sqrt{3}$. For $y = 0$ in (1), $x = -\frac{p}{m^2} = -3p$; hence the tangent cuts the axis of x, or axis of the curve at $(-3p, 0)$.

29. For what point of the parabola $y^2 = 4px$ is the length of the tangent equal to 4 times the abscissa of the point of contact?

The formula for the length of the tangent is

$$\text{Tan} = \sqrt{\text{subt}^2 + y_1^2} = \sqrt{4x_1^2 + 4px_1}.$$

Hence $\sqrt{4x_1^2 + 4px_1} = 4x_1$, or $x_1 = 0, \frac{1}{3}p$.

For $x_1 = \frac{1}{3}p$ in $y_1^2 = 4px_1$, $y_1 = \pm\frac{2}{3}p\sqrt{3}$; hence the required points are $(\frac{1}{3}p, \pm\frac{2}{3}p\sqrt{3})$.

30. The product of the tangent and normal is equal to twice the square of the ordinate of the point of contact. Find the point of contact and the inclination of the tangent to the axis of x.

By No. 29, $\tan = \sqrt{4x_1^2 + 4px_1}$.

And $\text{norm} = \sqrt{\text{subn}^2 + y_1^2} = \sqrt{4p^2 + 4px_1}$

Hence by the conditions of the problem, we have

$$\sqrt{4x_1^2 + 4px_1}\,\sqrt{4p^2 + 4px_1} = 2y_1^2 = 2(4px_1), \text{ or } x_1 = p.$$

For $x_1 = p$ in $y_1^2 = 4px_1$, $y_1 = \pm 2p$; hence the points of contact are $(p, \pm 2p)$. At $(p, 2p)$ the slope is $+1$, and at $(p, -2p)$ it is -1; hence the inclinations of the tangents are 45° and 135° respectively.

31. Two tangents to a parabola are perpendicular to each other. Find the product of their subtangents.

Let the points of contact be (x_1, y_1) and (x_2, y_2); then the slopes of the tangents are $\frac{2p}{y_1}$ and $\frac{2p}{y_2}$ respectively, and, since the tangents are perpendicular to each other, we have

$$\frac{2p}{y_1} \cdot \frac{2p}{y_2} = -1, \text{ or } y_1^2 y_2^2 = 16p^4;$$

hence $\qquad (4px_1)(4px_2) = 16p^4$, or $(2x_1)(2x_2) = 4p^2$,

in which $2x_1$ and $2x_2$ are the subtangents.

32. Prove that the circle described on a focal radius as diameter touches the tangent drawn through the vertex.

Let (x_1, y_1) be the point to which the focal radius r is drawn; then r, or the distance from $(p, 0)$ to (x_1, y_1), is $x_1 + p$, and $\frac{1}{2}(x_1 + p)$ is the abscissa of its middle point; hence the circle described on the focal radius as diameter touches the axes of y or the tangent at the vertex.

33. Prove that the circle described on a focal chord as diameter touches the directrix.

Let (x_1, y_1) and (x_2, y_2) be the extremities of any focal chord; then the half of the length of this chord is the half-sum of $x_1 + p$ and $x_2 + p$, or $p + \frac{1}{2}(x_1 + x_2)$. But $\frac{1}{2}(x_1 + x_2)$ is the abscissa of the middle point of the focal chord; hence $p + \frac{1}{2}(x_1 + x_2)$, or half the chord, is the distance of this point from the directrix; therefore the directrix is tangent to the circle whose diameter is a focal chord.

34. Find the locus of the middle points of all the ordinates of a parabola.

Let (x, y) be the middle point of the ordinate of the point (x_1, y_1) on $y^2 = 4px$; then $x_1 = x$, and $y_1 = 2y$. Substituting these values of x_1 and y_1 in $y_1^2 = 4px_1$, we obtain $y^2 = px$; hence the locus is a parabola whose latus rectum is p.

35. Find the locus of the middle points of all the focal radii.

Let (x, y) be the middle point of the focal radius to the point (x_1, y_1) on $y^2 = 4px$; then $x = \frac{1}{2}(p + x_1)$, and $y = \frac{1}{2}y_1$; or $x_1 = 2x - p$, and $y_1 = 2y$. Substituting these values of x_1 and y_1 in $y_1^2 = 4px_1$, we have as the equation of required locus

$$y^2 = 2p(x - \tfrac{1}{2}p). \qquad (1)$$

By changing the origin to $(\frac{1}{2}p, 0)$, (1) becomes $y^2 = 2px$; hence the locus is a parabola whose latus rectum is $2p$.

36. Find the locus of the middle points of all the focal chords.

If we take the focus as the origin, the equations of the parabola and the focal chords are

$$y^2 = 4p(x + p) \qquad (1)$$

and

$$y = mx. \qquad (2)$$

Eliminating y between (1) and (2), and solving for x, we have

$$x = \frac{2p}{m^2} \pm \frac{2p}{m^2}\sqrt{1 + m^2}, \qquad (3)$$

in which the two values of x are the abscissas of the extremities of the chord. Let (x_1, y_1) be its middle point; then from (3), [2], and (2), we obtain

$$x_1 = \frac{2p}{m^2} = 2p\frac{x^2}{y^2}. \qquad (4)$$

Since (x_1, y_1) is on (2), we have

$$y_1 = mx_1 = \frac{2p}{m} = 2p\frac{x}{y}, \text{ or } y_1^2 = 4p^2\frac{x^2}{y^2}. \qquad (5)$$

Dividing (5) by (4), we obtain as the equation of the required locus

$$y_1^2 = 2px_1, \text{ or } y^2 = 2px. \qquad (6)$$

Hence the locus is a parabola whose latus rectum is $2p$, and whose vertex is at the focus of the given parabola. When referred to parallel axes through the vertex of the given parabola, the equation of the curve (6) is evidently $y^2 = 2p(x - p)$.

37. Find the locus of the middle points of all chords passing through the vertex.

Let (x, y) be the middle point of the chord drawn from the vertex $(0, 0)$ to the point (x_1, y_1) on $y^2 = 4px$; then $x_1 = 2x$, $y_1 = 2y$.

Substituting these values in $y_1^2 = 4px_1$, we obtain as the equation of the required locus

$$4y^2 = 8px, \text{ or } y^2 = 2px.$$

38. Find the locus of the middle points of all chords that meet at the foot of the axis.

If we take the foot of the axis as the origin, the equations of the parabola and the chords are

$$y^2 = 4p(x - p), \tag{1}$$

and

$$y = mx. \tag{2}$$

Eliminating y between (1) and (2), and solving for x, we have

$$x = \frac{2p}{m^2} \pm \frac{2p}{m^2}\sqrt{1 - m^2}, \tag{3}$$

in which the two values of x are the abscissas of the extremities of the chord. Let (x_1, y_1) be its middle point; then from (3), [2], and (2), we obtain

$$x_1 = \frac{2p}{m^2} = 2p\frac{x^2}{y^2}. \tag{4}$$

Since (x_1, y_1) is on (2), we have

$$y_1 = mx_1 = \frac{2p}{m} = 2p\frac{x}{y}, \text{ or } y_1^2 = 4p^2\frac{x^2}{y^2}. \tag{5}$$

Dividing (5) by (4), we obtain as the equation of the required locus

$$y_1^2 = 2px_1, \text{ or } y^2 = 2px. \tag{6}$$

Hence the locus is a parabola whose latus rectum is $2p$, and whose vertex is at the foot of the axis of the given parabola. Its equation, when referred to parallel axes through the vertex of the given parabola, is evidently

$$y^2 = 2p(x + p).$$

The arc without the given parabola does not properly belong to the locus here required.

Two tangents to the parabola $y^2 = 4px$ make the angles θ, θ' with the axis of x; find the locus of their intersection.

39. If $\cot\theta + \cot\theta' = k$.

If the tangent $y = mx + \frac{p}{m}$ passes through (x', y'), we have

$$y' = mx' + \frac{p}{m}. \qquad (1)$$

Solving (1) for m, and denoting its two values by m_1 and m_2, we have

$$m_1 = \frac{y' + \sqrt{y'^2 - 4px'}}{2x'}, \qquad (2)$$

$$m_2 = \frac{y' - \sqrt{y'^2 - 4px'}}{2x'}. \qquad (3)$$

Hence $$m_1 + m_2 = \frac{y'}{x'}, \qquad (4)$$

and $$m_1 m_2 = \frac{p}{x'}. \qquad (5)$$

Here $$\cot\theta + \cot\theta' = \frac{1}{m_1} + \frac{1}{m_2} = k, \text{ or } m_1 + m_2 = k m_1 m_2. \qquad (6)$$

From (4), (5), and (6), we obtain

$$y' = kp, \text{ or } y = kp,$$

as the equation of the required locus, which is therefore a straight line parallel to the axis of x.

40. If $\cot\theta - \cot\theta' = k$.

Here $$\cot\theta - \cot\theta' = \frac{1}{m_1} - \frac{1}{m_2} = k, \text{ or } m_2 - m_1 = k m_1 m_2. \qquad (7)$$

From (7), and (2), (3), (5) of No. 39, we have

$$\frac{-\sqrt{y'^2 - 4px'}}{x'} = k\frac{p}{x_1}, \text{ or } y^2 - 4px = k^2p^2, \qquad (8)$$

as the equation of the required locus, which is therefore a parabola equal to the given parabola. Its vertex is the point $\left(-\frac{k^2p}{4}, 0\right)$, and its latus rectum is $4p$.

41. If $\tan\theta \tan\theta' = k$.

Here $$\tan\theta \tan\theta' = m_1 m_2 = k. \qquad (9)$$

From (9), and (5) of No. 39 we have as the equation of the required locus $kx = p$.

42. If $\sin\theta \sin\theta' = k$.

Here $\sin^2\theta \sin^2\theta' = \dfrac{1}{\csc^2\theta}\,\dfrac{1}{\csc^2\theta'} = k^2$, or $\csc^2\theta \csc^2\theta' = \dfrac{1}{k^2}$.

Hence, since $\tan\theta = m_1$ and $\tan\theta' = m_2$, we have

$$\frac{m_1^2+1}{m_1^2}\cdot\frac{m_2^2+1}{m_2^2} = \frac{1}{k^2},$$

or

$$\frac{(m_1m_2)^2 + m_1^2 + m_2^2 + 1}{(m_1m_2)^2} = \frac{1}{k^2}. \tag{10}$$

Substituting in (10) the values given in (2), (3), and (5) of No. 39, we obtain as the equation of the required locus

$$(x-p)^2 + y^2 = \frac{p^2}{k^2},$$

which is therefore a circle whose centre is $(p, 0)$ and whose radius is $p \div k$.

43. Find the locus of the centre of a circle which passes through a given point and touches a given straight line.

Take the given line as the axis of y and a perpendicular through the given point as the axis of x, and let the distance from the point to the line $= a$; then this point is $(a, 0)$. Let (x, y) be the centre of the circle; then the distance from (x, y) to $(a, 0)$ is equal to x; hence we have

$$(x-a)^2 + y^2 = x^2, \text{ or } y^2 = 2a\left(x - \tfrac{1}{2}a\right).$$

Hence the locus is a parabola whose vertex is $(\frac{1}{2}a, 0)$ and whose latus rectum is $2a$.

Ex. 31. Page 134.

1. Given a parabola, to draw its axis (§ 106).

Draw any two parallel chords PQ and $P'Q'$, bisect them in the points m and m'; then the line through m and m' will be a diameter. Draw any chord perpendicular to mm'; the line through its middle point parallel to mm' will be the axis of the curve.

2. Prove that the perpendicular dropped from any point of the directrix to the polar of the point passes through the focus.

The abscissa of any point on the directrix is $-p$: denote its ordinate by y_1. The polar of $(-p, y_1)$ with respect to the parabola $y^2=4px$ is

$$y_1y=2p(x-p). \qquad (1)$$

The perpendicular to (1) from $(-p, y_1)$ is

$$y-y_1=-\frac{y_1}{2p}(x+p). \qquad (2)$$

The co-ordinates of the focus $(p, 0)$ satisfy (2); hence line (2) passes through the focus.

Line (1) evidently passes through $(p, 0)$; hence the polar of any point on the directrix is a focal chord.

3. To find by construction the pole of a focal chord.

Through the focus draw a line perpendicular to the focal chord; the intersection of this line with the directrix will, by No. 2, be the pole of the focal chord.

4. Prove that through any point *three* normals can be drawn to a parabola.

Let the normal

$$y=mx-mp(2+m^2)$$

pass through the point (a, b); then we have

$$pm^3+(2p-a)m+b=0,$$

which, being a cubic, gives three different values for m.

Hence three normals can be drawn from (a, b) to $y^2=4px$.

5. Tangents are drawn through the ends of a chord. Prove that the part of the corresponding diameter contained between the chord and the intersection of the tangents is bisected by the curve.

Let the curve be referred to the corresponding diameter and the tangent at its vertex, which will be parallel to the chord, and denote one extremity of the chord by (x_1, y_1); then the other will be $(x_1, -y_1)$, and by § 114, Cor., the tangents are

$$\pm y_1y=2p'(x+x_1). \qquad (1)$$

For $y=0$ in (1), $x=-x_1$; hence these tangents intersect on this diameter, and the vertex of this diameter bisects the part of it between the chord and the intersection of the tangents.

6. Focal radii are drawn to two points of a parabola, and tangents are then drawn through these points. Prove that the angle between the tangents is equal to half the angle between the focal lines.

Let FP and FQ be any two focal radii, and AP and TQ tangents at P and Q; then $PFQ = 2\ TRA$.

Since FAP and FTQ are isosceles triangles,

$$\angle XFP = 2\angle FAP,$$

and $$\angle XFQ = 2\angle FTQ.$$

$$\therefore\ XFP - XFQ = 2(FAP - FTQ),$$

or $$\angle QFP = 2\angle ART.$$

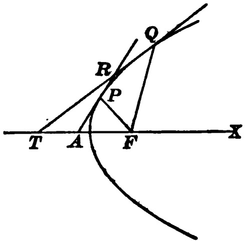

FIG. 67.

7. Show that if the vertex is taken as pole, the polar equation of a parabola is

$$\rho = \frac{4p\cos\theta}{\sin^2\theta}.$$

The principal equation of the parabola is $y^2 = 4px$.

Substituting $\rho\cos\theta$ for x, and $\rho\sin\theta$ for y, in (1), we obtain

$$\rho = \frac{4p\cos\theta}{\sin^2\theta}$$

as the polar equation of the parabola when the vertex is the pole.

8. Explain how tangents to a parabola may be drawn from an exterior point (§ 102).

Let H be the exterior point, $R'R$ the directrix, and F the focus. With H as a centre and HF as a radius describe a circle cutting the directrix in R and R'; through these points draw RP and $R'P'$ parallel to FX; then will HP and HP' be tangent to the curve at P and P'. For the triangles HPR and HPF being mutually equilateral are equal, and $\angle HPR = \angle HPF$; hence, by § 102, HP is a tangent. For like reason HP' is a tangent.

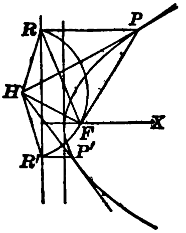

FIG. 68.

9. Having given a parabola, how would you find its axis, directrix, focus, and latus rectum?

Let *RAP* be the given parabola. Find the axis *AX* as in No. 1. Find the third proportional to *AM* and *MP*. By § 97, Cor., this third proportional will be the latus rectum. Take $AF = AD = \frac{1}{4}$ latus rectum; then *F* is the focus, and *DC*, a perpendicular to *DX*, is the directrix.

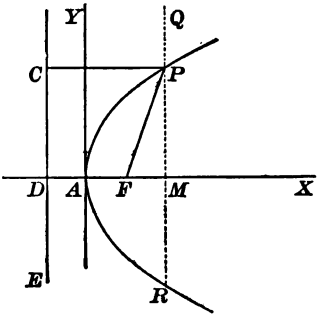

FIG. 69.

10. From the point $(-2, 5)$ tangents are drawn to the parabola $y^2 = 6x$. What is the equation of the chord of contact?

The equation of the chord of contact of tangents drawn from (h, k) to $y^2 = 4px$ is

$$ky = 2p(x + h). \tag{1}$$

Here $2p = 3$, and (h, k) is $(-2, 5)$. Substituting these values in (1), we obtain

$$5y = 3(x - 2), \text{ or } 3x - 5y - 6 = 0,$$

as the required equation.

11. The general equation of a system of parallel chords in the parabola $7y^2 = 25x$ is $4x - 7y + k = 0$. What is the equation of the corresponding diameter?

The equation of the diameter of $y^2 = 4px$, the slope of whose chords is m, is $y = \frac{2p}{m}$.

Here $2p = \frac{25}{14}$, and $m = \frac{4}{7}$; hence the equation of the diameter is

$$y = \tfrac{25}{14} \times \tfrac{7}{4} = \tfrac{25}{8}, \text{ or } 8y - 25 = 0.$$

12. In the parabola $y^2 = 13x$, what is the equation of the chords of the diameter $y + 11 = 0$?

Here $\frac{2p}{m} = -11$, and $2p = \frac{13}{2}$; hence m, the slope of the chords, is $-\frac{13}{22}$, and their equation is

$$13x + 22y + k = 0.$$

13. In the parabola $y^2 = 6x$, what chord is bisected at the point (4, 3)?

The chord is parallel to the tangent at the point whose ordinate is 3; hence its slope is $\frac{2p}{3}$, or 1. The chord through (4, 3) having the slope 1 is

$$y - 3 = x - 4, \text{ or } x - y - 1 = 0.$$

14. Given the parabola $y^2 = 4px$; find the equation of the chord which passes through the vertex and is bisected by the diameter $y = a$. How can this chord be constructed?

The equation of the chord through the vertex, or origin, is

$$y = mx, \tag{1}$$

and this chord is bisected by the diameter

$$y = \frac{2p}{m}. \tag{2}$$

Here $2p \div m = a$; hence $m = 2p \div a$. Substituting this value of m in (1), we obtain

$$ay = 2px$$

as the required equation.

To construct this chord, draw a line through the vertex parallel to the tangent at the vertex of the diameter $y = a$.

15. The latus rectum of a parabola = 16. What is the equation of the curve if a diameter at the distance 12 from the focus, and the tangent through its extremity, are taken as axes?

If the latus rectum is 16, the distance from the vertex to the focus is 4, and the principal equation of the parabola is $y^2 = 16x$.

For $y = 12$ in $y^2 = 16x$, $x = 9$; hence the new origin is (9, 12). The distance from the focus (4, 0) to (9, 12) is 13.

The required equation is of the form $y^2 = 4p'x$, in which $p' =$ the distance of the origin from the focus (§ 114); hence the equation is

$$y^2 = 52x.$$

16. Show that the equation of that chord of the parabola $y^2 = 4px$ which is bisected at the point (h, k) is

$$k(y - k) = 2p(x - h).$$

The slope of the chord is $2p \div k$, and the line through (h, k), having the slope $2p \div k$, is

$$k(y-k) = 2p(x-h).$$

17. Prove that the parameter of any diameter is equal to the focal chord of that diameter.

Let γ denote the angle between the axis of the curve and the focal chord bisected by any diameter; then by (2) of § 114 the parameter of this diameter is $4p \div \sin^2\gamma$.

If in $\rho = \dfrac{2p}{1-\cos\theta}$ we make $\theta = \gamma$ and $180° + \gamma$, we obtain as the length of the focal chord whose slope is $\tan\gamma$,

$$\text{Chord} = \frac{2p}{1-\cos\gamma} + \frac{2p}{1+\cos\gamma} = \frac{4p}{\sin^2\gamma},$$

which equals the parameter of the diameter bisecting this chord.

18. Prove that the locus of $y^2 - 6y - 8x + 25 = 0$ is a parabola; and determine the latus rectum, the vertex, the focus, the axis, and the directrix.

The equation $y^2 - 6y - 8x + 25 = 0$ may be written in the form

$$(y-3)^2 = 8(x-2). \qquad (1)$$

If we change the reference of (1) to parallel axes whose origin is (2, 3), we obtain

$$y^2 = 8x. \qquad (2)$$

Hence the locus of (1) is a right-hand X-parabola whose latus rectum is 8, whose vertex is (2, 3), whose focus is (4, 3), since it is p, or 2, to the right of the vertex. The axis is $y = 3$, or the new axis of x; the directrix is $x = 0$, since it is 2 to the left of the vertex.

19. Prove that in general the locus of $y^2 + Ax + By + C = 0$ is a parabola; and determine its latus rectum, its vertex, and its axis.

The equation $y^2 + Ax + By + C = 0$ may be written in the form

$$\left(y + \frac{B}{2}\right)^2 + A\left(x - \frac{B^2 - 4C}{4A}\right) = 0. \qquad (1)$$

If we change the reference of (1) to parallel axes, whose origin is $\left(\dfrac{B^2-4C}{4A},\ -\dfrac{B}{2}\right)$, we obtain

$$y^2 = -Ax. \qquad (2)$$

Hence the locus of (1) is a parabola whose latus rectum is numerically A, and whose vertex is $\left(\frac{B^2-4C}{4A},\ -\frac{B}{2}\right)$.

Its axis is $y=-\frac{1}{2}B$, and the parabola is a right or left hand parabola according as A is negative or positive.

20. Prove that in general the locus of $x^2+Ax+By+C=0$ is a parabola; and determine its latus rectum, its vertex, and its axis.

The given equation may be written in the form

$$\left(x+\frac{A}{2}\right)^2+B\left(y-\frac{A^2-4C}{4B}\right)=0. \tag{1}$$

If we change the reference of (1) to parallel axes whose origin is $\left(-\frac{A}{2},\ \frac{A^2-4C}{4B}\right)$, we obtain

$$x^2=-By. \tag{2}$$

Hence the locus of (1) is a parabola whose latus rectum is numerically B and whose vertex is $\left(-\frac{A}{2},\ \frac{A^2-4C}{4B}\right)$. Its axis is $x=-\frac{A}{2}$, and the parabola is an upward or downward Y-parabola, according as B is negative or positive.

21. Find the locus of the centre of a circle which touches a given circle and also a given straight line.

Take the given straight line as the axis of y, and a perpendicular to it through the centre of the given circle as the axis of x. Let the radius of the circle $=r$, and the distance from the centre to the given line $=a$.

Let (x, y) be the centre of the circle; then the distance of (x, y) from the centre of the given circle $(a, 0)$, diminished or increased by the radius r, equals the distance of (x, y) from the given line, or the axis of y; that is,

$$\sqrt{(x-a)^2+y^2}\mp r=x,$$

or $$(x-a)^2+y^2=x^2\pm 2rx+r^2,$$

or $$y^2=2(a\pm r)x+r^2-a^2,$$

is the required equation.

22. The area and base of a triangle being given, find the locus of the intersection of perpendiculars dropped from the ends of the base to the opposite sides.

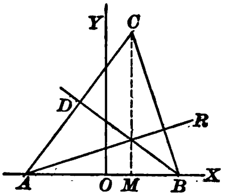

FIG. 70.

Let ABC be a triangle whose base $AB = 2a$, and whose area $= ab$; then $MC = b$. Take AX as the axis of x, and O the middle point of the base as the origin; then A is $(-a, 0)$, B is $(a, 0)$, and if we denote OM by c, C is (c, b).

The slope of AC, or the line through $(-a, 0)$ and (c, b), is $\frac{b}{c+a}$; hence the equation of the line perpendicular to AC through B $(a, 0)$ is

$$y = -\frac{c+a}{b}(x-a). \tag{1}$$

The slope of BC, or the line through $(a, 0)$ and (c, b), is $\frac{b}{c-a}$; hence the equation of the line perpendicular to BC through A $(-a, 0)$ is

$$y = -\frac{c-a}{b}(x+a). \tag{2}$$

Combining (1) and (2) so as to eliminate c, we obtain as the required equation

$$x^2 + by = a^2, \text{ or } x^2 = -b\left(y - \frac{a^2}{b}\right). \tag{3}$$

If we change the reference of (3) to parallel axes through $\left(0, \frac{a^2}{b}\right)$, we obtain $x^2 = -by$, whose locus is a downward Y-parabola whose latus rectum is b.

Ex. 32. Page 144.

1. What are a, b, c, and e in the ellipse whose equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1\,? \tag{1}$$

Equation (1) is in the form of [30]; hence $a^2 = 25$, $b^2 = 16$, $c^2 = a^2 - b^2 = 25 - 16 = 9$. $\therefore a = 5, b = 4, c = 3$, and $e = \frac{c}{a} = \frac{3}{5}$.

2. What are a, b, c, and e in the ellipse whose equation is

$$x^2 + 2y^2 = 2? \tag{1}$$

Writing (1) in the form of [30], we have

$$\frac{x^2}{2} + \frac{y^2}{1} = 1;$$

hence $a^2 = 2$, $b^2 = 1$, $c^2 = a^2 - b^2 = 2 - 1 = 1$; $\therefore a = \sqrt{2}$, $b = 1$, $c = 1$, and $e = c \div a = \frac{1}{2}\sqrt{2}$.

3. What are a, b, c, and e in the ellipse whose equation is

$$3x^2 + 4y^2 = 12? \tag{1}$$

Writing (1) in the form of [30], we have

$$\frac{x^2}{4} + \frac{y^2}{3} = 1;$$

hence $a^2 = 4$, $b^2 = 3$, $c^2 = a^2 - b^2 = 1$; $\therefore a = 2$, $b = \sqrt{3}$, $c = 1$, $e = \frac{1}{2}$.

4. What are a, b, c, and e in the ellipse whose equation is

$$Ax^2 + By^2 = 1? \tag{1}$$

Equation (1) may be written in the form

$$\frac{x^2}{1 \div A} + \frac{y^2}{1 \div B} = 1. \tag{2}$$

(i.) If $A < B$, (2) is in the form of [30]; hence $a^2 = 1 \div A$, $b^2 = 1 \div B$, $c^2 = (1 \div A) - (1 \div B)$; $\therefore a = \sqrt{\frac{1}{A}}$, $b = \sqrt{\frac{1}{B}}$, $c = \sqrt{\frac{B - A}{AB}}$, $e = \sqrt{\frac{B - A}{B}}$.

(ii.) If $A = B$, the locus of (2) is a circle whose radius is $\sqrt{\frac{1}{A}}$.

(iii.) If $A > B$, (2) is in the form of (10), § 119; hence $a^2 = 1 \div B$, $b^2 = 1 \div A$, $c^2 = (1 \div B) - (1 \div A)$; $\therefore a = \sqrt{\frac{1}{B}}$, $b = \sqrt{\frac{1}{A}}$, $c = \sqrt{\frac{A - B}{AB}}$, $e = \sqrt{\frac{A - B}{A}}$.

5. Find the latus rectum of the ellipse $3x^2 + 7y^2 = 18$.

By § 125, the latus rectum $= 2b^2 \div a$.

Here $a = \sqrt{6}$, and $b^2 = \frac{18}{7}$; $\therefore$ latus rectum $= \frac{6}{7}\sqrt{6}$.

6. Find the eccentricity of an ellipse if its latus rectum is equal to one-half its minor axis.

Here latus rectum $= 2b^2 \div a = b$, or $b = \frac{1}{2}a$.

But $c^2 = a^2 - b^2$; $\therefore\ c^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2$, or $c = \frac{1}{2}a\sqrt{3}$;

hence $e = c \div a = \frac{1}{2}a\sqrt{3} \div a = \frac{1}{2}\sqrt{3}$.

7. What is the equation of an ellipse if the axes are 12 and 8?

Here $a = 6$, and $b = 4$; hence, by [30], the equation is

$$\frac{x^2}{36} + \frac{y^2}{16} = 1, \text{ or } 4x^2 + 9y^2 = 144.$$

8. What is the equation of an ellipse if major axis $= 26$, distance between foci $= 24$?

Here $a = 13$, $c = 12$; $\therefore\ b^2 = a^2 - c^2 = 25$, or $b = 5$; hence the required equation is

$$\frac{x^2}{169} + \frac{y^2}{25} = 1, \text{ or } 25x^2 + 169y^2 = 4225.$$

9. What is the equation of an ellipse if sum of axes $= 54$, distance between foci $= 18$?

Here $c = 9$ and $a + b = 27$. (1)

But $a^2 - b^2 = c^2 = 81$. (2)

(2) $\div$ (1) gives $a - b = 3$. (3)

From (1) and (3) we obtain $a = 15$, $b = 12$; hence the required equation is

$$144x^2 + 225y^2 = 32{,}400.$$

10. What is the equation of an ellipse if latus rectum $= \frac{64}{5}$, eccentricity $= \frac{3}{5}$?

Here latus rectum $= \frac{2b^2}{a} = \frac{64}{5}$, and eccentricity $= \frac{c}{a} = \frac{3}{5}$.

Hence $b^2 = \frac{32}{5}a$, $c = \frac{3}{5}a$. Substituting these values in $a^2 = b^2 + c^2$, we find $a = 10$; hence $b = 8$, and the required equation is

$$16x^2 + 25y^2 = 1600.$$

11. What is the equation of an ellipse if minor axis $= 10$, distance from focus to vertex $= 1$?

Here $b = 5$, and $c = a - 1$. Substituting these values in $a^2 = b^2 + c^2$, we find $a = 13$; hence the required equation is

$$25x^2 + 169y^2 = 4225.$$

12. What is the equation of an ellipse if the curve passes through $(1, 4)$ and $(-6, 1)$?

Since the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through $(1, 4)$ and $(-6, 1)$, we have

$$\frac{1}{a^2} + \frac{16}{b^2} = 1 \text{ and } \frac{36}{a^2} + \frac{1}{b^2} = 1. \qquad (1)$$

Solving equations (1) for a^2 and b^2, we find $a^2 = \frac{115}{3}$, $b^2 = \frac{115}{7}$. Substituting these values in [30], we obtain

$$3x^2 + 7y^2 = 115.$$

13. What is the equation of an ellipse if major axis $= 20$, minor axis $=$ distance between foci?

Here $a = 10$, and $b = c$. Substituting these values in $a^2 = b^2 + c^2$, we have $100 = 2b^2$, or $b^2 = 50$. Substituting the values of a^2 and b^2 in [30], we obtain after reduction $x^2 + 2y^2 = 100$.

14. What is the equation of an ellipse if sum of the focal radii of a point in the curve $= 3$ times the distance between the foci?

Sum of focal radii $= 2a$; hence $2a = 3(2c)$, or $a^2 = 9c^2$. Substituting this value of a^2 in $a^2 = b^2 + c^2$, we obtain $b^2 = 8c^2$. Substituting these values of a^2 and b^2 in [30], we have

$$8x^2 + 9y^2 = 72c^2 = 8a^2.$$

15. Prove that the semi-minor axis is a mean proportional between the segments of the major axis made by one of the foci.

Since $b^2 = a^2 - c^2 = (a + c)(a - c)$, we have

$$a + c : b :: b : a - c. \qquad \text{Q.E.D.}$$

16. What is the ratio of the two axes if the centre and foci divide the major axis into four equal parts?

From the condition $c = \frac{1}{2}a$; $\therefore b^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2$, or $2b = a\sqrt{3}$.

Hence $a : b :: 2 : \sqrt{3}$.

17. For what point of an ellipse is the abscissa equal to the ordinate?

If $x = y$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $x = y = \pm\frac{ab}{\sqrt{a^2+b^2}}$; hence the points are $\left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right)$ and $\left(\frac{-ab}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$.

There is a point in the second quadrant and one in the fourth whose co-ordinates are numerically equal.

18. Find the intersections of the loci $3x^2 + 6y^2 = 11$ and $y = x + 1$.

Regarding these equations as simultaneous, and solving, we find $x = \frac{1}{3}, -\frac{5}{3}$, and $y = \frac{4}{3}, -\frac{2}{3}$; hence the required points are $(\frac{1}{3}, \frac{4}{3})$ and $(-\frac{5}{3}, -\frac{2}{3})$.

19. Find the intersections of the loci $2x^2 + 3y^2 = 14$ and $y^2 = 4x$.

Regarding these equations as simultaneous, and solving, we find $x = 1, -7$, and $y = \pm 2, \pm 2\sqrt{-7}$; hence the points of intersection are $(1, 2)$ and $(1, -2)$. The pairs of values $(-7, 2\sqrt{-7})$ and $(-7, -2\sqrt{-7})$ do not give points of intersection of the plane loci of these equations, since one of each is imaginary.

20. Find the intersections of the loci $x^2 + 7y^2 = 16$ and $x^2 + y^2 = 10$.

Regarding these equations as simultaneous, and solving, we find $x = 3, -3$; $y = \pm 1, \pm 1$; hence the points of intersection are $(3, 1)$, $(3, -1)$, $(-3, 1)$, $(-3, -1)$.

21. The ordinates of the circle $x^2 + y^2 = r^2$ are bisected; find the locus of the points of bisection.

Let (x_1, y_1) be any point on the required locus; then $(x_1, 2y_1)$ will be the corresponding point on the circle. Substituting x_1 for x, and $2y_1$ for y, in $x^2 + y^2 = r^2$, we have

$$x_1^2 + 4y_1^2 = r^2, \text{ or } x^2 + 4y^2 = r^2,$$

as the required locus.

22. A straight line AB so moves that the points A and B always touch two fixed perpendicular straight lines. Show that any point P in AB describes an ellipse, and find its equation.

Let OB and OA be the two fixed perpendiculars, and AB the line of constant length, in which P is any point. Take OB and OA as the axes; let $AP = a$, $PB = b$, and $OBA = \phi$; then $x = a \cos\phi$, $y = b \sin\phi$. (1)

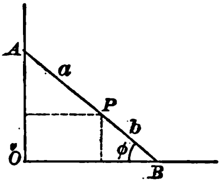

FIG. 71.

Equations (1) being of the form of (1) in § 128, Cor., they are the equations of an ellipse whose semi-axes are a and b. Or (1) is readily reduced to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

23. What is the locus of $Ax^2 + By^2 = C$ when C is zero? When is this locus imaginary?

When $C = 0$, the equation may be written in the form $y = \pm x\sqrt{-\frac{A}{B}}$, of which the locus is two straight lines through the origin when A and B have unlike signs. When A and B have like signs, the locus is imaginary except the point $(0, 0)$.

24. Prove that the abscissas of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are to the corresponding abscissas of the minor auxiliary circle $x^2 + y^2 = b^2$, as $a : b$.

Let x and x' represent the abscissas of points in the ellipse and the minor auxiliary circle respectively, corresponding to the same ordinate y; then from the equations of the two curves we have

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}, \quad x' = \pm\sqrt{b^2 - y^2}.$$

Whence $x : x' :: a : b$.

25. Construct an ellipse by the method of § 128.

Construct any number of points as in § 128, and trace the curve through them.

26. Construct an ellipse, having given c and b.

Construct a right triangle whose sides are c and b; then the hypotenuse will be a, and we have the axis to construct the curve.

27. Construct the axes of an ellipse, having given the foci and one point of the curve.

Bisect in O the line joining the foci F and F'. Draw the focal radii of the point P, and from O lay off $\frac{1}{2}(FP + F'P)$ in the direction of each focus. We thus obtain the vertices A and A'. Draw a perpendicular to AA' at O, and with F as a centre and OA as a radius, strike an arc cutting this perpendicular in B and B'; then AA' and BB' will be the major and minor axes respectively, and the curve is readily constructed

28. Construct the minor axis and foci, having given the major axis (in magnitude and position) and one point of the curve.

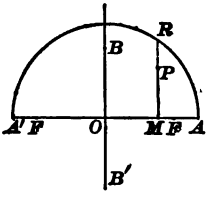

FIG. 72.

Let AA' be the major axis and P the point. Through O the middle point of $A'A$ draw the perpendicular BB'. From the equation of the ellipse we obtain

$$\sqrt{(a+x)(a-x)} : y :: a : b, \qquad (1)$$

in which $a + x = A'M$, and $a - x = MA$. On $A'A$ as diameter describe the semi-circumference ABA', produce MP to meet the semi-circumference as in R; then $MR = \sqrt{(a+x)(a-x)}$.
Lay off $OB = OB'$ so that $MR : MP :: OA : OB$; then BB' is the minor axis. With B as a centre and OA as a radius strike an arc cutting AA'; the points of intersection F and F' will be the foci required.

29. A square is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the equations of the sides and the area of the square.

If (x, y) be a vertex, then $x = y$. Whence we have

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1, \text{ or } x = \pm \frac{ab}{\sqrt{a^2 + b^2}}; \text{ also, } y = \pm \frac{ab}{\sqrt{a^2 + b^2}}.$$

$$\text{Area} = \frac{2ab}{\sqrt{a^2 + b^2}} \cdot \frac{2ab}{\sqrt{a^2 + b^2}} = \frac{4a^2b^2}{a^2 + b^2}.$$

Ex. 33. Page 151.

1. What are the equations of the tangent and the normal to the ellipse $2x^2 + 3y^2 = 35$ at the points whose abscissa $= 2$?

The equations of the tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) are respectively

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1, \qquad [32]$$

and
$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1). \qquad [33]$$

Here $a^2 = \frac{35}{2}$, $b^2 = \frac{35}{3}$, and, since for $x = 2$ in $2x^2 + 3y^2 = 35$, $y = \pm 3$, (x_1, y_1) is $(2, 3)$ and $(2, -3)$. Substituting these values in [32] and [33], we obtain as the required equations,

$$4x \pm 9y = 35,$$

and
$$9x \mp 4y = 6.$$

2. What are the equations of the tangent and the normal to the ellipse $4x^2 + 9y^2 = 36$ at the points whose abscissa $= -\frac{3}{2}$?

Here $a^2 = 9$, $b^2 = 4$, and, since for $x = -\frac{3}{2}$, $y = \pm\sqrt{3}$, (x_1, y_1) is $(-\frac{3}{2}, \sqrt{3})$ and $(-\frac{3}{2}, -\sqrt{3})$. Substituting these values in [32] and [33], we obtain as the required equations,

$$\frac{-\frac{3}{2}x}{9} + \frac{\pm y\sqrt{3}}{4} = 1, \text{ or } 2x \mp 3y\sqrt{3} + 12 = 0,$$

and
$$y \mp \sqrt{3} = \frac{9(\pm\sqrt{3})}{4(-\frac{3}{2})}(x + \tfrac{3}{2}), \text{ or } 6\sqrt{3}x \mp 4y + 5\sqrt{3} = 0.$$

3. Find the equations of the tangent and the normal to the ellipse $x^2 + 4y^2 = 20$ at the point of contact $(2, 2)$. Also find the subtangent and the subnormal.

Reducing this equation to the form of [30], we have

$$\frac{x^2}{20} + \frac{y^2}{5} = 1.$$

Hence $a^2 = 20$, $b^2 = 5$, and (x_1, y_1) is $(2, 2)$. Substituting these values in [32] and [33], we have

$$\frac{2x}{20} + \frac{2y}{5} = 1, \text{ or } x + 4y = 10, \qquad (1)$$

and
$$y - 2 = \tfrac{40}{10}(x - 2), \text{ or } 4x - y - 6 = 0. \qquad (2)$$

For $y = 0$ in (1), $x = 10 =$ intercept.
$\therefore$ Subt $[= x_1 -$ intercept$] = 2 - 10 = -8$.
For $y = 0$ in (2), $x = \frac{3}{2} =$ intercept.
$\therefore$ Subn $=$ intercept $- x_1 = \frac{3}{2} - 2 = -\frac{1}{2}$.

4. Show that the line $y = x + \sqrt{\frac{5}{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$.

Regarding these equations as simultaneous, and solving for x and y, we find that their only common point is $(-\frac{1}{10}\sqrt{30}, \frac{1}{15}\sqrt{30})$; hence the line $y = x + \sqrt{\frac{5}{6}}$ is a tangent at that point.

Or, substituting the values of a^2 and b^2 in [36], we obtain as the equation of any tangent to $2x^2 + 3x^2 = 1$,

$$y = mx + \sqrt{\tfrac{1}{2}m^2 + \tfrac{1}{3}}.$$

Putting $m = 1$ in (1), we obtain

$$y = x + \sqrt{\tfrac{5}{6}};$$

hence the given equation is a tangent.

5. Required the condition which must be satisfied in order that the straight line $\frac{x}{m} + \frac{y}{n} = 1$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(x_1\ y_1)$ is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$. (1)

Writing $\frac{x}{m} + \frac{y}{n} = 1$ in the form of (1), we have

$$\frac{(a^2 \div m)\,x}{a^2} + \frac{(b^2 \div n)\,y}{b^2} = 1. \qquad (2)$$

Now if (2) is a tangent to the ellipse, the point $\left(\frac{a^2}{m}, \frac{b^2}{n}\right)$ must be on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; hence we have

$$\frac{a^4 \div m^2}{a^2} + \frac{b^4 \div n^2}{b^2} = 1, \text{ or } \frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

Or, writing $\frac{x}{m} + \frac{y}{n} = 1$ in the form of $y = mx + \sqrt{a^2m^2 + b^2}$, we have

$$y = -\frac{n}{m}x + n. \qquad (1)$$

Hence, if (1) is a tangent,

$$m = -\frac{n}{m}, \text{ and } \sqrt{a^2m^2 + b^2} = \sqrt{a^2\frac{n^2}{m^2} + b^2} = n.$$

$$\therefore \frac{a^2n^2}{m^2} + b^2 = n^2, \text{ or } \frac{a^2}{m^2} + \frac{b^2}{n^2} = 1,$$

is the required condition.

6. In an ellipse the subtangent for the point $(3, \frac{12}{5})$ is $-\frac{16}{3}$, the eccentricity $= \frac{4}{5}$. What is the equation of the ellipse?

Here $\text{subt} = \frac{x_1{}^2 - a^2}{x_1} = \frac{9 - a^2}{3} = -\frac{16}{3}$, or $a^2 = 25$.

$$\frac{c}{a} = \frac{4}{5}, \text{ or } c^2 = \tfrac{16}{25}a^2 = 16.$$

$$b^2 = a^2 - c^2 = 25 - 16 = 9.$$

Hence the equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1, \text{ or } 9x^2 + 25y^2 = 225.$$

7. What is the equation of a tangent to the ellipse $9x^2 + 64y^2 = 576$ parallel to the line $2y = x$?

Here $a^2 = 64$ and $b^2 = 9$; hence, by [36], the equation of any tangent to $9x^2 + 64y^2 = 576$ is

$$y = mx \pm \sqrt{64m^2 + 9}. \qquad (1)$$

If (1) is parallel to $2y = x$, $m = \frac{1}{2}$. Substituting this value of m in (1), we obtain as the required equations,

$$y = \tfrac{1}{2}x \pm 5, \text{ or } 2y = x \pm 10.$$

8. Find the equation of a tangent to the ellipse $3x^2 + 5y^2 = 15$ parallel to the line $4x - 3y - 1 = 0$.

Here $a^2 = 5$ and $b^2 = 3$; hence, by [36], the equation of any tangent to the given ellipse is

$$y = mx \pm \sqrt{5m^2 + 3}. \qquad (1)$$

If (1) is parallel to $4x - 3y - 1 = 0$, $m = \frac{4}{3}$. Substituting this value of m in (1), we obtain as the required equations

$$y = \tfrac{4}{3}x \pm \tfrac{1}{3}\sqrt{107}, \text{ or } 3y = 4x \pm \sqrt{107}.$$

Or, the tangent to $\frac{x^2}{5}+\frac{y^2}{3}=1$ is

$$\frac{x_1x}{5}+\frac{y_1y}{3}=1. \tag{1}$$

The slope of (1) is $\frac{-3x_1}{5y_1}$; hence if (1) is parallel to $4x-3y-1=0$ we have

$$\frac{-3x_1}{5y_1}=\frac{4}{3} \tag{2}$$

Since (x_1, y_1) is on the curve, we have

$$3x_1^2+5y_1^2=15. \tag{3}$$

Solving (2) and (3), we find the points of contact to be

$$\left(\frac{-20}{\sqrt{107}}, \frac{9}{\sqrt{107}}\right) \text{ and } \left(\frac{20}{\sqrt{107}}, \frac{-9}{\sqrt{107}}\right).$$

Substituting these values of x_1 and y_1 in (1), we obtain as the required equations

$$4x-3y\pm\sqrt{107}=0.$$

9. In what points do the tangents which are equally inclined to the axes touch the ellipse $b^2x^2+a^2y^2=a^2b^2$?

If the tangents are equally inclined to the axis, their slopes must be ± 1; hence we have

$$\frac{-b^2x_1}{a^2y_1}=\pm 1. \tag{1}$$

Since (x_1, y_1) is on the ellipse, we have

$$b^2x_1^2+a^2y_1^2=a^2b^2. \tag{2}$$

Solving (1) and (2), we find that the point of contact is

$$\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right) \text{ or } \left(\frac{-a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right).$$

10. Through what point of the ellipse $b^2x^2+a^2y^2=a^2b^2$ must a tangent and a normal be drawn in order that they may form, with the axis of x as base, an isosceles triangle?

Since the tangent and normal are perpendicular to each other, if they form an isosceles triangle with the axis of x, the slope of the tangent must be ± 1, and the solution and the answers are the same as that of No. 9.

11. Through a point of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and the corresponding point of the auxiliary circle $x^2 + y^2 = a^2$, normals are drawn. What is the ratio of the subnormals?

In the circle subn $= -x_1$.

In the ellipse subn $= -\frac{b^2}{a^2}x_1$.

Hence subn in ellipse : subn in circle :: $b^2 : a^2$.

12. For what points of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the subtangent equal numerically to the abscissa of the point of contact?

Here we have subt $= \frac{x_1^2 - a^2}{x_1} = -x_1$, or $x_1 = \pm \frac{1}{2}a\sqrt{2}$. Substituting x_1 and y_1 for x and y in the given equation, and solving, we find $y_1 = \pm \frac{1}{2}b\sqrt{2}$; hence the points are $(\frac{1}{2}a\sqrt{2}, \pm \frac{1}{2}b\sqrt{2})$ and $(-\frac{1}{2}a\sqrt{2}, \pm \frac{1}{2}b\sqrt{2})$.

13. Find the equations of tangents drawn from the point (3, 4) to the ellipse $16x^2 + 25y^2 = 400$.

Here $a^2 = 25$, and $b^2 = 16$; hence by [36] the equation of any tangent to the given ellipse is

$$y = mx + \sqrt{25m^2 + 16}. \tag{1}$$

If (1) passes through (3, 4), we have

$$4 - 3m = \sqrt{25m^2 + 16}. \tag{2}$$

Solving (2) for m, we obtain $m = 0$ and $-\frac{3}{2}$. Substituting these values of m in (1), we obtain as the required equations

$$y = 4, \text{ and } 3x + 2y = 17.$$

14. What are the equations of the tangents drawn through the extremities of the latera recta of the ellipse $4x^2 + 9y^2 = 36a^2$?

The extremities of the latera recta are (c, ± ½ latus rectum) and ($-c$, ± ½ latus rectum). Here $a = 3a$, $b = 2a$; hence $c = \sqrt{a^2 - b^2} = a\sqrt{5}$; also ½ latus rectum $= \frac{b^2}{a} = \frac{4a}{3}$ (§ 125). Hence the points of contact are $\left(a\sqrt{5}, \pm\frac{4a}{3}\right)$ and $\left(-a\sqrt{5}, \pm\frac{4a}{3}\right)$. Substituting these values of x_1 and y_1 in the equation of the tangent to $4x^2 + 9y^2 = 36a^2$ at (x_1, y_1); that is, in

$$4x_1x + 9y_1y = 36a^2,$$

we obtain as the required equations

$$\sqrt{5}x \pm 3y = 9a, \quad -\sqrt{5}x \pm 3y = 9a.$$

15. What is the distance from the centre of an ellipse to a tangent making the angle ϕ with the major axis?

The distance from the origin to the tangent

$$y = mx + \sqrt{a^2m^2 + b^2}$$

is
$$\sqrt{\frac{a^2m^2 + b^2}{m^2 + 1}} = a\sqrt{\frac{m^2 + 1 - \frac{a^2 - b^2}{a^2}}{m^2 + 1}}$$

$$= a\sqrt{1 - \frac{e^2}{m^2 + 1}} = a\sqrt{1 - e^2\cos^2\phi},$$

since $m = \tan\phi$.

16. What is the area of the triangle formed by the tangent in the last problem and the axes of co-ordinates?

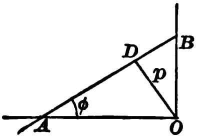

FIG. 73.

We could find the equation of the tangent, obtain its intercepts, and take one-half their product. Or, let AB (Fig. 73) be the tangent, OD the perpendicular on it from O; then

$$\angle BOD = \angle OAD = \phi,$$
$$OA = p \div \sin\phi,$$
$$OB = p \div \cos\phi;$$

hence
$$\text{Area} = \tfrac{1}{2}(OA \times OB) = \tfrac{1}{2}\frac{p^2}{\sin\phi\cos\phi} = \tfrac{1}{2}\frac{a^2(1 - e^2\cos^2\phi)}{\sin\phi\cos\phi}$$
$$= \tfrac{1}{2}(a^2\csc\phi\sec\phi - c^2\cot\phi).$$

17. From the point where the auxiliary circle cuts the minor axis produced tangents are drawn to the ellipse. Find the points of contact.

If the tangent $b^2x_1x + a^2y_1y = a^2b^2$ passes through $(0, a)$ or $(0, -a)$, we have $\pm a^3y_1 = a^2b^2$, or $y_1 = \pm(b^2 \div a) =$ semi-latus rectum. Hence the points of contact are the extremities of the latera recta.

18. Prove that the tangents drawn through the ends of a chord through the centre are parallel.

Let (x_1, y_1) be one end of the chord; then from the symmetry of the curve we know that $(-x_1, -y_1)$ is the other end. The slope of the tangent at (x_1, y_1) is $\frac{-b^2x_1}{a^2y_1}$, and at $(-x_1, -y_1)$ is $\frac{-b^2(-x_1)}{a^2(-y_1)}$, or $\frac{-b^2x_1}{a^2y_1}$. As these slopes are equal, the tangents are parallel.

19. Find the locus of the foot of a perpendicular dropped from the focus to a tangent.

The equation of the tangent at any point (x_1, y_1) is

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad (1)$$

The line from the focus $(ae, 0)$ perpendicular to (1) is

$$y = -\frac{1}{m}(x - ae). \qquad (2)$$

From (1), $\qquad y - mx = \sqrt{a^2m^2 + b^2}. \qquad (3)$

From (2), $\qquad my + x = ae. \qquad (4)$

Adding the squares of (3) and (4), and dividing by $1 + m^2$, we obtain

$$x^2 + y^2 = a^2$$

as the equation of the required locus; hence the locus is the circle whose diameter is the major axis.

Ex. 34. Page 152.

1. Given the ellipse $36x^2 + 100y^2 = 3600$. Find the equations and the lengths of focal radii drawn to the point $(8, \frac{18}{5})$.

If r and r' are the focal radii to the point (x, y) of the ellipse, then $r = a - ex$, and $r' = a + ex$.

Here $x = 8$, $a = 10$, and $b = 6$; hence $ae = \sqrt{a^2 - b^2} = 8$; or $e = \frac{4}{5}$.

Substituting these values in (1), we obtain

$$r = \tfrac{18}{5}, \quad r' = \tfrac{82}{5}. \qquad (1)$$

The line through $(8, \frac{18}{5})$ and the focus $(8, 0)$ is evidently parallel to the axis of y and 8 to the right of it; hence its equation is

$$x = 8. \qquad (2)$$

The line through $(8, \frac{18}{5})$ and the focus $(-8, 0)$ is

$$\frac{y}{x + 8} = \frac{\frac{18}{5}}{8 + 8}, \text{ or } 40y = 9x + 72. \qquad (3)$$

If we had used [4] to obtain (2), we would have obtained $\dfrac{y}{x - 8} = \dfrac{\frac{18}{5}}{0}$, which is satisfied only when $x - 8 = 0$; hence $x = 8$ is the required equation.

2. Is the point $(2, 1)$ within or without the ellipse $2x^2 + 3y^2 = 12$?

The point (h, k) is without, on, or within the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ according as $\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} - 1 >, =, \text{ or } < 0.$

Hence, since $\frac{2^2}{6}+\frac{1^2}{4}-1<0$, the point (2, 1) is within the ellipse $\frac{x^2}{6}+\frac{y^2}{4}=1$.

3. Find the eccentricity of an ellipse if the equation is

$$2x^2+3y^2=12.$$

Here $a=\sqrt{6}$, $b=2$; hence $ae=\sqrt{a^2-b^2}=\sqrt{2}$, or $e=\frac{\sqrt{2}}{\sqrt{6}}=\frac{1}{3}\sqrt{3}$.

4. Find the eccentricity of an ellipse if the angle $FBF'=90°$ (see Fig. 74).

If $FBF'=90°$, $OBF'=45°=OF'B$; hence $OF'=OB$ or $c=b$, and $a=\sqrt{b^2+c^2}=b\sqrt{2}$. Hence $e=\frac{c}{a}=\frac{b}{b\sqrt{2}}=\frac{1}{2}\sqrt{2}$.

5. Find the eccentricity of an ellipse if LFR is the latus rectum and LOR is an equilateral triangle (F being the focus, O the centre).

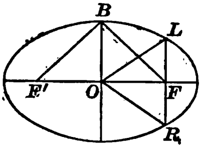

Fig. 74.

If $\triangle\, LOR$ is equilateral, $\angle\, LOR=60°$; hence $\angle\, LOF=30°$.

$$OF=ae,\quad OL=2\,FL=\frac{2b^2}{a}.$$

Hence $$\cos FOL=\cos 30°=\frac{OF}{OL}=\frac{a^2e}{2b^2},$$

or $$e=\sqrt{3}\,\frac{b^2}{a^2}. \qquad (1)$$

But $$e^2=\frac{a^2-b^2}{a^2}. \qquad (2)$$

From (1) and (2) we have

$$a^4-b^2a^2=3b^4,\ \text{or}\ a^2=\frac{b^2+b^2\sqrt{13}}{2}. \qquad (3)$$

The minus sign before $b^2\sqrt{13}$ is excluded since $a^2>b^2$.

From (3) and (1) we have

$$e=\frac{2\sqrt{3}}{1+\sqrt{13}}.$$

6. Find the equations of tangents to an ellipse if they make equal intercepts on the axes.

If the intercepts are equal, the tangent must cross the first or the third quadrant; and hence its slope must be -1, and we have

$$\frac{-b^2x_1}{a^2y_1} = -1, \text{ or } b^2x_1 = a^2y_1. \qquad (1)$$

But (x_1, y_1) is on the curve; hence

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \text{ or } b^2x_1^2 + a^2y_1^2 = a^2b^2. \qquad (2)$$

Solving (1) and (2) for x_1 and y_1, we find the points of contact to be

$$\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right) \text{ and } \left(\frac{-a^2}{\sqrt{a^2+b^2}}, \frac{-b^2}{\sqrt{a^2+b^2}}\right).$$

Substituting these values of x_1 and y_1 in [32], we obtain

$$x + y = \pm\sqrt{a^2+b^2} \qquad (3)$$

as the required equations.

Or substituting -1 for m in

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \qquad [36]$$

we obtain (3) as the required equations.

7. Find the equations of tangents to an ellipse if they are parallel to BF (Fig. 74).

If the tangents are parallel to BF, their slope must be $-(b \div c)$, and we have

$$\frac{-b^2x_1}{a^2y_1} = -\frac{b}{c}, \text{ or } bx_1 = \frac{a^2y_1}{c}. \qquad (1)$$

But (x_1, y_1) is on the curve; hence

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \text{ or } b^2x_1^2 + a^2y_1^2 = a^2b^2. \qquad (2)$$

Solving (1) and (2) for x_1 and y_1, we find the points of contact to be

$$\left(\frac{a^2}{\sqrt{a^2+c^2}}, \frac{bc}{\sqrt{a^2+c^2}}\right) \text{ and } \left(\frac{-a^2}{\sqrt{a^2+c^2}}, \frac{-bc}{\sqrt{a^2+c^2}}\right).$$

Substituting these values of x_1 and y_1 in [32], we obtain

$$bx + cy = \pm b\sqrt{a^2+c^2} \qquad (3)$$

as the required equations.

Or substituting $-(b \div c)$ for m in

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \qquad [36]$$

we obtain (3) as the required equations.

8. Find the equations of tangents to an ellipse which are parallel to the line $\frac{x}{a}+\frac{y}{b}=1$ (a and b being the semi-axes).

Here $m=-(b\div a)$. Substituting this value of m in

$$y=mx\pm\sqrt{a^2m^2+b^2}, \qquad [36]$$

and reducing, we obtain

$$bx+ay=\pm ab\sqrt{2}$$

as the required equations.

9. Find the equation of a tangent in terms of the eccentric angle ϕ of the point of contact.

From § 128, Cor., we have $x_1=a\cos\phi$, $y_1=b\sin\phi$. Substituting these values of x_1 and y_1 in [32], we obtain

$$\frac{ax\cos\phi}{a^2}+\frac{by\sin\phi}{b^2}=1, \text{ or } \frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1.$$

10. Find the distance from the centre of an ellipse to a tangent through the point of contact (x_1, y_1).

Reducing [32] to the normal form, we obtain

$$p=\frac{a^2b^2}{\sqrt{a^4y_1^2+b^4x_1^2}}=\frac{ab}{\sqrt{\frac{a^2}{b^2}y_1^2+\frac{b^2}{a^2}x_1^2}}$$

$$=\frac{ab}{\sqrt{a^2-x_1^2+\frac{b^2}{a^2}x_1^2}}=\frac{ab}{\sqrt{a^2-e^2x_1^2}}.$$

11. Find the distance from the centre of an ellipse to a tangent making the angle ϕ with the axis of x.

See No. 15, Ex. 33.

12. In what ratio is the abscissa of a point divided by the normal at that point?

In Fig. 75, $ON=\frac{c^2}{a^2}x_1$, $NM=\frac{b^2}{a^2}x_1$ numerically; hence

$$ON:NM::c^2:b^2.$$

13. At the point (x_1, y_1) of an ellipse a normal is drawn. What is the product of the segments into which it divides the major axis?

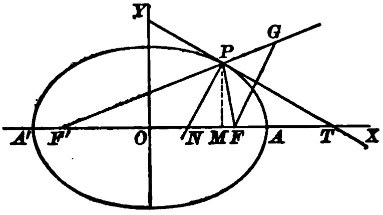

FIG. 75.

Let P be (x_1, y_1); then

$$\begin{aligned} A'N &= A'O + ON \\ &= a + e^2x_1. \\ NA &= OA - ON \\ &= a - e^2x_1. \end{aligned}$$

Hence

$$\begin{aligned} &A'N \times NA \\ &= (a + e^2x_1)(a - e^2x_1) \\ &= a^2 - e^4x_1^2. \end{aligned}$$

14. Find the length of PN (Fig. 75).

$$\begin{aligned} PN^2 = MN^2 + MP^2 &= \left(\frac{b^2}{a^2}x_1\right)^2 + y_1^2 = \frac{b^4}{a^4}x_1^2 + b^2 - \frac{b^2}{a^2}x_1^2 \\ &= b^2 - \frac{b^2}{a^2}\left(1 - \frac{b^2}{a^2}\right)x_1^2 = a^2(1 - e^2) - (1 - e^2)e^2x_1^2 \\ &= (a^2 - e^2x_1^2)(1 - e^2). \end{aligned}$$

Whence $PN = \sqrt{(1 - e^2)(a^2 - e^2x_1^2)}$.

15. Determine the value of the eccentric angle at the end of the latus rectum.

The end of the latus rectum in the first quadrant is $\left(ae, \dfrac{b^2}{a}\right)$;

hence $ae = a\cos\phi$ or $\cos\phi = e$,

and $b\sin\phi = \dfrac{b^2}{a}$, or $\sin\phi = \dfrac{b}{a} = \sqrt{\dfrac{b^2}{a^2}} = \sqrt{1 - e^2}$.

Whence $\tan\phi = \dfrac{\sqrt{1 - e^2}}{e} = \dfrac{\sqrt{a^2 - a^2e^2}}{ae} = \dfrac{b}{c}$.

16. Prove that the semi-minor axis b of an ellipse is a mean proportional between the distances from the foci to a tangent.

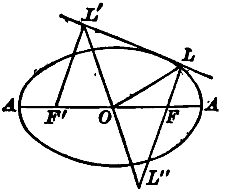

FIG. 76.

By No. 19, Ex. 33, we have

$$OL = OL' = a.$$

Now

$$\begin{aligned} &OF = OF', \\ &\angle F'OL' = \angle FOL'', \\ &\angle OF'L' = \angle OFL''; \end{aligned}$$

hence $OL'' = OL' = a$, and LL'' is a chord of the auxiliary circle.

Whence

$$\begin{aligned} FL \times F'L' &= FL \times FL'' = FA \times FA' \\ &= (a - ae)(a + ae) = a^2 - a^2e^2 = b^2. \end{aligned}$$

17. Prove that the semi-minor axis b of an ellipse is a mean proportional between a normal and the distance from the centre to the corresponding tangent.

By No. 14, normal $= \sqrt{(1-e^2)(a^2-e^2x_1^2)}$.

By No. 10, distance to tangent $= p = \dfrac{ab}{\sqrt{a^2-e^2x_1^2}}$.

Hence $p(\text{normal}) = ab\sqrt{1-e^2} = b\sqrt{a^2-a^2e^2} = b^2$.

18. Determine and describe the locus of the middle point of that portion of a tangent contained between the tangents drawn through the vertices.

The minor axis produced is parallel to the tangents at the vertices and midway between them; hence it bisects all lines included between these tangents.

19. Determine and describe the locus of the middle point of a perpendicular dropped from a point of the circle $(x-a)^2+y^2=r^2$ to the axis of y.

Let (x, y) be any point on the circle $(x-a)^2+y^2=r^2$, and (x', y') be the middle point of the perpendicular from (x, y) to the axis of y; then $x=2x'$ and $y=y'$. Substituting these values in $(x-a)^2+y^2=r^2$, we obtain as the equation of the required locus

$$(2x'-a)^2+y'^2=r^2, \text{ or } 4(x-\tfrac{1}{2}a)^2+y^2=r^2. \qquad (1)$$

Changing the reference of (1) to new parallel axes with $(\frac{1}{2}a, 0)$ as the origin, we have

$$4x^2+y^2=r^2, \text{ or } \frac{x^2}{\frac{1}{4}r^2}+\frac{y^2}{r^2}=1. \qquad (2)$$

The locus of (2) is an ellipse whose centre is the new origin $(\frac{1}{2}a, 0)$, and whose semi-axes are r and $\frac{1}{2}r$, the transverse axis lying on the axis of y.

20. Determine and describe the locus of the middle point of a chord of the ellipse $b^2x^2+a^2y^2=a^2b^2$ drawn through the positive end of the minor axis.

Let (x, y) be any point on the ellipse, and (x', y') the middle point of the chord drawn from $(0, b)$ to (x, y); then

$$x'=\tfrac{1}{2}x, \text{ or } x=2x',$$

and $$y'=\tfrac{1}{2}(y+b), \text{ or } y=2y'-b.$$

Substituting these values of x and y in the equation of the ellipse, we have as the equation of the required locus,

$$4b^2x'^2 + a^2(2y' - b)^2 = a^2b^2, \text{ or } 4b^2x^2 + 4a^2(y - \tfrac{1}{2}b)^2 = a^2b^2. \quad (1)$$

Changing the origin to $(0, \frac{1}{2}b)$, (1) becomes

$$4b^2x^2 + 4a^2y^2 = a^2b^2, \text{ or } \frac{x^2}{\frac{1}{4}a^2} + \frac{y^2}{\frac{1}{4}b^2} = 1. \quad (2)$$

The locus of (2) is an ellipse whose semi-axes are $\frac{1}{2}a$ and $\frac{1}{2}b$; and whose centre is at the new origin $(0, \frac{1}{2}b)$.

21. Determine and describe the locus of the vertex of a triangle whose base $2c$ and sum of the other sides $2s$ are given.

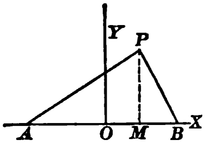

FIG. 77.

Let ABP be a triangle in which $AB = 2c$ and $AP + PB = 2s$. Take AB as the axis of x, and its middle point O as the origin. Denote P as (x, y); then $OM = x$, $AM = c + x$, $MB = c - x$, $MP = y$.

From the right triangles AMP and BMP, we have

$$AP = \sqrt{AM^2 + MP^2} = \sqrt{(c + x)^2 + y^2}. \quad (1)$$

$$PB = \sqrt{MB^2 + MP^2} = \sqrt{(c - x)^2 + y^2}. \quad (2)$$

Hence $\sqrt{(c + x)^2 + y^2} = 2s - \sqrt{(c - x)^2 + y^2}$. (3)

Squaring (3),

$$s\sqrt{(c - x)^2 + y^2} = s^2 - cx,$$

which is readily reduced to the form

$$\frac{x^2}{s^2} + \frac{y^2}{s^2 - c^2} = 1. \quad (4)$$

The locus of (4) is an ellipse whose centre is the origin, and whose semi major and minor axes are s and $\sqrt{s^2 - c^2}$ respectively.

22. Determine and describe the locus of the vertex of a triangle, having given the base $2c$ and the product k of the tangents of the angles at the base.

Using the same notation and figure as in No. 21, we have

$$\tan MAP = \frac{y}{c + x}, \quad \tan MBP = \frac{y}{c - x}.$$

Whence $k = \dfrac{y}{c + x} \cdot \dfrac{y}{c - x} = \dfrac{y^2}{c^2 - x^2}$,

or $\dfrac{x^2}{c^2} + \dfrac{y^2}{kc^2} = 1.$ (1)

Here k and c^2 are both positive.

If $k = 1$, the locus of (1) is a circle whose centre is the origin, and whose radius is c. In this case APB is a right angle; hence the locus of the vertex of the right angle of a right triangle whose hypotenuse is AB is the circumference of a circle whose diameter is AB.

For other values of k, the locus of (1) is an ellipse, whose semi-axes are c and $c\sqrt{k}$, of which c always lies on the axis of x. The transverse axis lie on the axis of x, or that of y, according as $k <$ or > 1.

23. Determine and describe the locus of the symmetrical point of the right-hand focus of an ellipse with respect to a tangent.

In Fig. 61, § 140, let FH be a perpendicular from F to any tangent, as PQ at Q. Produce the focal radius $F'Q$ and the perpendicular FH until they intersect. Their point of intersection G is the symmetrical point of F with respect to the tangent TQ. For in the triangles FHQ and GHQ, QH is common, and $\angle HQG = \angle HQF$, since each equals $\angle F'QK$. Hence $FH = HG$.

Since $FQ = QG$, $F'G = F'Q + QF = 2a$, and G is in the circumference of a circle whose centre is F', and whose radius is $2a$. Its equation is

$$(x + c)^2 + y^2 = 4a^2.$$

Ex. 35. Page 164.

1. Find the area of the ellipse $x^2 + 4y^2 = 16$.

$$\text{Area} = \pi ab.$$

Here $a = 4$, and $b = 2$; hence area $= 8\pi$.

2. Find the distances of the directrices from the centre, in Ex. 1.

The distance of each directrix from the centre is $a \div e$.

Here $a = 4$, $b = 2$, and $e^2 = \frac{a^2 - b^2}{a^2} = \frac{16 - 4}{16} = \frac{12}{16}$, or $e = \frac{1}{2}\sqrt{3}$.

Hence $a \div e = 4 \div \frac{1}{2}\sqrt{3} = \frac{8}{3}\sqrt{3}$.

3. What is the equation of the polar of the point (5, 7) with respect to the ellipse $4x^2 + 9y^2 = 36$?

The equation of the polar of the point (h, k) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$; hence the equation of the polar of the point (h, k) with respect to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is $\frac{hx}{9}+\frac{ky}{4}=1$, which, when (h, k) is $(5, 7)$, becomes

$$\tfrac{5}{9}x+\tfrac{7}{4}y=1, \text{ or } 20x+63y=36.$$

4. Prove that a focal chord is perpendicular to the line which joins its pole to the focus. In what line does the pole lie?

Here the polar passes through the focus $(ae, 0)$, and from its equation we obtain $h=a \div e$; hence the pole of any focal chord is in the directrix, and is $\left(\frac{a}{e}, k\right)$.

Hence the equation of any focal chord, or the polar of $\left(\frac{a}{e}, k\right)$, is

$$\frac{x}{ae}+\frac{ky}{b^2}=1, \text{ or } y=-\frac{b^2x}{aek}+\frac{b^2}{k}. \qquad (1)$$

But the slope of the line from $\left(\frac{a}{e}, k\right)$ to $(ae, 0)$ is $\frac{aek}{b^2}$; hence this line is perpendicular to (1).

5. Find the pole of the line $Ax+By+C=0$ with respect to the ellipse $b^2x^2+a^2y^2=a^2b^2$.

The pole of the polar $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } (h, k).$$

Reducing $Ax+By+C=0$ to the form of $\frac{hx}{a^2}+\frac{ky}{b^2}=1$, we have

$$\frac{-(A \div C)a^2x}{a^2}-\frac{(B \div C)b^2y}{b^2}=1,$$

in which the pole (h, k) is evidently $\left(\frac{-Aa^2}{C}, \frac{-Bb^2}{C}\right)$.

6. Each of the two tangents which can be drawn to an ellipse from any point on its directrix subtends a right angle at the focus.

The chord of contact, or polar, of any point on the directrix $\left(\frac{a}{e}, k\right)$ is

$$\frac{x}{ae}+\frac{ky}{b^2}=1. \qquad (1)$$

But (1) passes through $(ae, 0)$, or is a focal chord; hence, by No. 4, each tangent subtends a right angle.

7. The two tangents which can be drawn to an ellipse from any external point subtend equal angles at the focus.

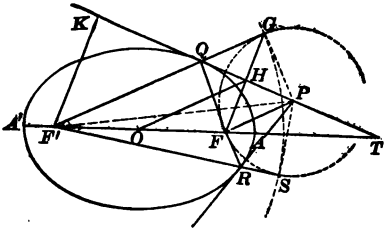

FIG. 78.

PQ and PR are tangents drawn from P to the ellipse.

Triangles $F'PG$ and $F'PS$ are equal, since $F'G = F'S$, $PG = PS$, $F'P = F'P$; hence $\angle PF'Q = \angle PF'R$. Again, triangles PQG and PQF are equal, and hence $\angle PGQ = \angle PFQ$; also triangles PRF and PRS are equal, and hence $\angle PFR = \angle PSR$; hence, as

$$\angle PGQ = \angle PSR, \quad \angle PFR = \angle PFQ.$$

No. 6 is but a particular case of 7.

8. Find the slope m_1 of a diameter if the square of the diameter is (i.) an arithmetic, (ii.) a geometric, (iii.) an harmonic mean between the squares of the axes.

The square of any diameter $= 4(b^2 + e^2x_1^2)$, (§ 145), in which x_1 is the abscissa of its extremity.

(i.) Here $4(b^2 + e^2x_1^2) = 2(a^2 + b^2)$, or $x_1^2 = \frac{1}{2}a^2$. (1)

From (1) and $a^2y_1^2 + b^2x_1^2 = a^2b^2$ we obtain

$$y_1^2 = \tfrac{1}{2}b^2. \tag{2}$$

The equation of any diameter of an ellipse is

$$y = -\frac{b^2x}{a^2m}, \tag{3}$$

in which m is the slope of its chords. Since (x_1, y_1) is on (3), from (1), (2), and (3) we obtain

$$m^2 = \frac{b^2}{a^2}.$$

Now $$m_1^2 = \frac{b^4}{a^4m^2} = \frac{b^4}{a^4} \cdot \frac{a^2}{b^2} = \frac{b^2}{a^2}.$$

(ii.) Here $4(b^2 + e^2x_1^2) = 4ab$, or $x_1^2 = \frac{ab - b^2}{e^2}$. (4)

From (4) and $a^2y_1^2 + b^2x_1^2 = a^2b^2$,

$$y_1^2 = \frac{b^2(a-b)}{ae^2}. \qquad (5)$$

But (x_1, y_1) is on (3); hence from (4), (5), and (3), we obtain

$$m^2 = \frac{b^3}{a^3}.$$

But $$m_1^2 = \frac{b^4}{a^4m^2} = \frac{b^4}{a^4} \cdot \frac{a^3}{b^3} = \frac{b}{a}.$$

(iii.) Here the three quantities are $4a^2$, $4(b^2 + e^2x_1^2)$, and $4b^2$, of which $4(b^2 + e^2x_1^2)$ is the harmonic mean; hence

$$4a^2 - 4(b^2 + e^2x_1^2) : 4(b^2 + e^2x_1^2) - 4b^2 :: 4a^2 : 4b^2,$$

or $$a^2e^2 - e^2x_1^2 : e^2x_1^2 :: a^2 : b^2, \text{ or } x_1^2 = \frac{a^2b^2}{a^2 + b^2}. \qquad (6)$$

From (6) and $a^2y_1^2 + b^2x_1^2 = a^2b^2$, we obtain

$$y_1^2 = \frac{a^2b^2}{a^2 + b^2}. \qquad (7)$$

From (6), (7), and (3) we obtain

$$m^2 = \frac{b^4}{a^4}; \therefore m_1^2 = 1.$$

9. Given the length $2l$ of a diameter, its inclination θ to the axis, and the eccentricity; find the major and minor axes.

From § 145, $l^2 = b^2 + e^2x_1^2$.

$$x_1 = l\cos\theta; \therefore b^2 = l^2 - e^2l^2\cos^2\theta,$$

or $$b = l\sqrt{1 - e^2\cos^2\theta}.$$

$$a^2 = \frac{b^2}{1 - e^2} = \frac{l^2(1 - e^2\cos^2\theta)}{1 - e^2},$$

or $$a = l\sqrt{\frac{1 - e^2\cos^2\theta}{1 - e^2}}.$$

10. Tangents at the extremities of a chord of the ellipse intersect on the diameter which bisects that chord.

Take as the axis of x the diameter which bisects the chord, and as the axis of y the diameter parallel to the chord; then if (x_1, y_1) is one extremity of the chord, $(x_1, -y_1)$ is the other. These diameters, or axes of reference, being conjugate diameters, the equations of the tangents at (x_1, y_1) and $(x_1, -y_1)$ are respectively

$$\frac{x_1x}{a'^2}+\frac{y_1y}{b'^2}=1, \tag{1}$$

and

$$\frac{x_1x}{a'^2}-\frac{y_1y}{b'^2}=1. \tag{2}$$

Now the intercepts of (1) and (2) on the axis of x are equal; hence they intersect on this axis, or the diameter bisecting the chord.

11. Tangents are drawn from (3, 2) to the ellipse $x^2+4y^2=4$. Find the equation of the chord of contact, and of the line which joins (3, 2) to the middle point of the chord.

The equation of the chord of contact of tangents from (h, k) to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{hx}{a^2}+\frac{ky}{b^2}=1. \tag{1}$$

Hence the chord of contact of tangents from (3, 2) to

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \text{ is } \frac{3x}{4}+\frac{2y}{1}=1, \text{ or } 3x+8y=4. \tag{2}$$

By No. 10 the line through (3, 2) and the middle point of the chord is a diameter through (3, 2); hence its equation is

$$y=\tfrac{2}{3}x, \text{ or } 2x-3y=0.$$

12. Find the area of the rectangle whose sides are the two segments into which a focal chord is divided by the focus.

If, in [39], we put $\theta=\theta'$ and $\theta=180^\circ+\theta'$, the two values of ρ will be the segments of the focal chord whose inclination to the transverse axis is θ. Hence the sum of the two segments equals

$$\frac{a(1-e^2)}{1-e\cos\theta'}+\frac{a(1-e^2)}{1+e\cos\theta'}, \text{ or } \frac{2a(1-e^2)}{1-e^2\cos^2\theta'};$$

and the product of the segments equals

$$\frac{a^2(1-e^2)^2}{1-e^2\cos^2\theta'}=\frac{a(1-e^2)}{2}\cdot\frac{2a(1-e^2)}{1-e^2\cos^2\theta'}=\frac{b^2}{2a}\cdot\frac{2a(1-e^2)}{1-e^2\cos^2\theta'}.$$

13. What is the equation of a chord in the ellipse $13x^2 + 11y^2 = 143$ which passes through $(1, 2)$ and is bisected by the diameter $3x - 2y = 0$?

Reducing the equation of the ellipse to the symmetrical form, we have

$$\frac{x^2}{11} + \frac{y^2}{13} = 1. \tag{1}$$

From (1) we see that the transverse axis lies on the axis of y; hence the equation of the diameter which bisects chords whose slope is m is

$$y = -\frac{a^2x}{b^2m}. \tag{2}$$

Here $a^2 = 13$, $b^2 = 11$. Substituting these values in (2), we have

$$y = -\frac{13x}{11m}. \tag{3}$$

Comparing $3x - 2y = 0$ and (3), we have

$$\frac{-13}{11m} = \frac{3}{2}, \text{ or } m = -\tfrac{26}{33}.$$

The chord through (1, 2), having the slope $-\tfrac{26}{33}$, is

$$y - 2 = -\tfrac{26}{33}(x - 1), \text{ or } 26x + 33y = 92.$$

14. In the ellipse $9x^2 + 36y^2 = 324$ find the equation of a chord passing through (4, 2) and bisected at this point.

The equation of a diameter to $b^2x^2 + a^2y^2 = a^2b^2$ is

$$y = -\frac{b^2x}{a^2m}. \tag{1}$$

Here $a^2 = 36$, $b^2 = 9$; substituting these values in (1), we have

$$y = -\frac{x}{4m}. \tag{2}$$

If (2) passes through (4, 2), we have

$$2 = -\frac{1}{m}, \text{ or } m = -\tfrac{1}{2}.$$

Hence the slope of the chord that is bisected at (4, 2) is $-\tfrac{1}{2}$, and its equation is $x + 2y = 8$.

15. Write the equations of diameters conjugate to the following lines:

$$x - y = 0,\ x + y = 0,\ ax = by,\ ay = bx.$$

The equation of condition of conjugate diameters is

$$mm' = -\frac{b^2}{a^2},$$

in which m and m' are their slopes.

If $y = x$ is one diameter, $m = 1$; hence $m' = -\frac{b^2}{a^2}$, and the conjugate diameter is $a^2y + b^2x = 0$.

If $y = -x$ is one diameter, $m = -1$; hence $m' = \frac{b^2}{a^2}$, and the conjugate diameter is $b^2x - a^2y = 0$.

If $by = ax$ is one diameter, $m = a \div b$; hence $m' = -\frac{b^3}{a^3}$, and the conjugate diameter is $a^3y + b^3x = 0$.

If $ay = bx$ is one diameter, $m = b \div a$; hence $m' = -\frac{b}{a}$, and the conjugate diameter is $ay + bx = 0$.

16. Show that the lines $2x - y = 0$, $x + 3y = 0$ are conjugate diameters in the ellipse $2x^2 + 3y^2 = 4$.

Here $a^2 = 2$, $b^2 = \frac{4}{3}$; hence the equation of condition of conjugate diameters in $2x^2 + 3y^2 = 4$ is

$$mm' = -\tfrac{2}{3}. \tag{1}$$

The lines $2x - y = 0$ and $x + 3y = 0$ are both diameters, and their slopes satisfy (1); hence they are conjugate diameters.

17. Find the equation of a diameter parallel to the normal at the point (x_1, y_1), the semi-axes being a and b.

The slope of the normal at (x_1, y_1) is $\frac{a^2y_1}{b^2x_1}$; hence the equation of the diameter parallel to this normal is

$$y = \frac{a^2y_1}{b^2x_1}x, \text{ or } b^2x_1y = a^2y_1x.$$

18. The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-minor axis.

Let d and d' denote the length of the perpendiculars.

The equation of the tangent at (x_1, y_1) is

$$a^2y_1y + b^2x_1x = a^2b^2.$$

By [12] we have

$$d = \frac{b^2x_1ae - a^2b^2}{\sqrt{a^4y_1^2 + b^4x_1^2}} = \frac{-b(a - ex_1)}{\sqrt{\frac{a^2}{b^2}y_1^2 + \frac{b^2}{a^2}x_1^2}}$$

$$= -\frac{b}{b'}(a - ex_1). \qquad (\S\ 145)$$

In like manner, we obtain

$$d' = -\frac{b}{b'}(a + ex_1).$$

Hence $$dd' = \frac{b^2}{b'^2}(a^2 - e^2x_1^2) = b^2. \qquad (\S\ 145)$$

19. The diagonals of the parallelogram *RKEH* are conjugate diameters.

For they are evidently parallel to the supplementary chords drawn from *D* to the diameter *SCS'*.

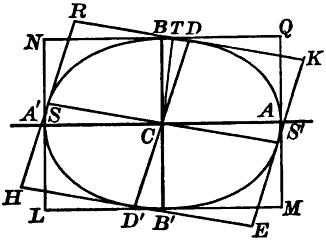

FIG. 79.

20. The angle between two semi-conjugate diameters is a maximum, when they are equal.

Since $a'^2 + b'^2 = a^2 + b^2$, that is, is constant for all values for a' and b', the product $a'^2b'^2$, or $a'b'$, is greatest when $a'^2 = b'^2$, or $a' = b'$; that is, when the semi-diameters are equal. But when $a'b'$ is greatest, $\sin\theta$ or $ab \div a'b'$ is least, and the obtuse angle θ is greatest.

21. The eccentric angles corresponding to equal semi-conjugate diameters are 45° and 135°.

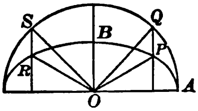

FIG. 80.

By § 148, Cor. 2, the equal conjugate diameters lie on the diagonals of the rectangle on the axes of the ellipse; so that if *OP* and *OR* were equal semi-conjugate diameters, *OP* and *OR*, as also *OQ* and *OS*, would make equal angles with *OB*; hence $QOB = SOB = 45°$ (§ 146). $\therefore AOQ = 45°$, and $AOS = 135°$.

22. The polar of a point in a diameter is parallel to the conjugate diameter.

The polar of the point (h, k) with respect to the ellipse

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1 \text{ is } \frac{hx}{a'^2}+\frac{ky}{b'^2}=1. \qquad (1)$$

If the point (h, k) be on the axis of x, (1) becomes

$$hx = a'^2,$$

whose locus is parallel to the axis of y.

If (h, k) is on the axis of y, (1) becomes $ky = b'^2$.

23. Find the equations of equal conjugate diameters.

Since the equal conjugate diameters lie on the diagonals of the rectangle on the axes, their equations are

$$y = \pm\frac{b}{a}x.$$

24. The length of a semi-diameter is l; find the equation of the conjugate diameter.

By § 145 we have

$$l^2 = b^2 + e^2x_1^2, \text{ or } x_1 = \frac{\pm\sqrt{l^2-b^2}}{e}. \qquad (1)$$

From (1) and the equation of the ellipse we obtain

$$y_1 = \frac{b}{ae}\sqrt{a^2-l^2}. \qquad (2)$$

In (1) and (2), (x_1, y_1) denotes the upper extremity of the diameter. From (4) in § 144, and (1) and (2), we have

$$x_2 = -\frac{a}{b}y_1 = -\frac{1}{e}\sqrt{a^2-l^2}, \qquad (3)$$

$$y_2 = \frac{b}{a}x_1 = \pm\frac{b}{ae}\sqrt{l^2-b^2}. \qquad (4)$$

Hence the equation of the conjugate diameter is

$$y = \frac{y_2}{x_2}x = \mp\frac{b\sqrt{l^2-b^2}}{a\sqrt{a^2-l^2}}x,$$

or

$$bx\sqrt{l^2-b^2} \pm ay\sqrt{a^2-l^2} = 0.$$

25. The angle between two equal conjugate diameters = 120°; find the eccentricity of the ellipse.

The equal conjugate diameters lie on the diagonals of the rectangle on the axes; hence, if their included angle is 120°, the angle between one of them and the transverse axes is 30°, and we have

$$\frac{b}{a} = \tan 30^\circ = \tfrac{1}{3}\sqrt{3}, \text{ or } b = \tfrac{1}{3}\sqrt{3}\,a.$$

Hence $$e^2 = \frac{a^2 - b^2}{a^2} = \frac{a^2 - \frac{1}{3}a^2}{a^2}, \text{ or } e = \tfrac{1}{3}\sqrt{6}.$$

26. Given a diameter, to construct the conjugate diameter.

Draw a chord parallel to the given diameter, and a diameter through one extremity of the chord. Draw the supplemental chord; the diameter parallel to it will be the conjugate to the given diameter.

27. To draw a tangent to a given ellipse parallel to a given straight line.

Through one end of the transverse axis draw a chord parallel to the given line, draw the supplementary chord and a diameter parallel to it; the tangents at the extremities of this diameter will be parallel to the given line by § 143.

28. Given an ellipse; to find by construction the centre, foci, and axes.

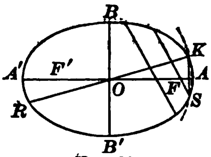

FIG. 31.

Given the ellipse $A'BAB'$; draw any two parallel chords, and through their middle points draw KOR. The middle point of KR, which is a diameter, is the centre. With O as a centre, and OK as a radius, draw an arc, and let K and S be two of its intersections with the ellipse. Draw the chord KS, which will be a chord of one of the axes, since its ends are equally distant from the centre O. The two diameters, one perpendicular to KS, and the other parallel to it, will be the axes required, the longer, AA', being the major axis. With B as a centre, and OA as a radius, strike an arc cutting the transverse axis: the points of intersection F' and F will be the foci.

29. Find the rectangular equation of the ellipse, taking the origin at the left-hand vertex.

Referred to the centre, the new origin will be $(-a, 0)$. Substituting $x-a$ for x in [30], we obtain

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a}, \qquad (1)$$

as the equation required.

30. Find the polar equation of an ellipse, taking as pole the right-hand focus.

From § 119 we have

$$\rho = a - ex,$$

in which $x = ae + \rho\cos\theta$.

Hence $\rho = a - ae^2 - \rho e\cos\theta$, or $\rho = \dfrac{a(1-e^2)}{1+e\cos\theta}$.

31. Find the polar equation of the ellipse, taking the centre as pole.

Substituting $\rho\cos\theta$ for x, and $\rho\sin\theta$ for y in [30], we obtain

$$\frac{\rho^2\cos^2\theta}{a^2}+\frac{\rho^2\sin^2\theta}{b^2}=1.$$

Substituting $1-\cos^2\theta$ for $\sin^2\theta$, and remembering that $\dfrac{a^2-b^2}{a^2}=e^2$, we obtain $\rho^2 = \dfrac{b^2}{1-e^2\cos^2\theta}$.

32. If the centre of an ellipse is the point (4, 7), and the major and minor axes are 14 and 8, find its equation, the axes being supposed parallel to the axes of co-ordinates.

The equation of the ellipse referred to its one centre is

$$\frac{x^2}{49}+\frac{y^2}{16}=1, \qquad (1)$$

Since the centre is (4, 7), the origin referred to the axes of the ellipse is $(-4, -7)$. Substituting in (1) $x-4$ for x, and $y-7$ for y, we obtain $\dfrac{(x-4)^2}{49}+\dfrac{(y-7)^2}{16}=1,$

or $16x^2+49y^2-128x-686y+1873=0,$

as the equation required.

33. The equation of an ellipse, the origin being at the left-hand vertex, is $25x^2 + 81y^2 = 450x$; find the axes.

Writing this equation in the form of (1), No. 29, we obtain

$$\frac{x^2}{81} + \frac{y^2}{25} = \frac{2x}{9}.$$

Whence $a = 9$ and $b = 5$.

34. If the minor axis $= 12$, and the latus rectum $= 5$, what is the equation of the ellipse, the origin being taken at the left-hand vertex?

Here $b = 6$, and $\dfrac{2b^2}{a} = 5$, or $a = \tfrac{72}{5}$.

Substituting these values in (1) of No. 29, we obtain after reduction

$$\frac{25x^2}{144} + y^2 = 5x.$$

35. Find the eccentric angle ϕ corresponding to the upper end of the diameter whose length is $2c$.

From § 145 we have

$$c^2 = b^2 + e^2x_1^2, \text{ or } x_1^2 = \frac{c^2 - b^2}{e^2}. \quad (1)$$

By § 128 and (1) we have

$$\cos^2\phi = \frac{x_1^2}{a^2} = \frac{c^2 - b^2}{a^2e^2} = \frac{c^2 - b^2}{a^2 - b^2},$$

or

$$\cos\phi = \sqrt{\frac{c^2 - b^2}{a^2 - b^2}}.$$

36. At the intersection of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = ab$ tangents are drawn to both curves. Find the angle between them.

From the symmetry of the curves it is sufficient to consider the intersection in the first quadrant. Regarding the equations as simultaneous, we find for the intersection in the first quadrant,

$$x = \frac{1}{e}\sqrt{ab - b^2}, \quad y = \frac{b}{ae}\sqrt{a^2 - ab}.$$

The slope of the circle at this point is

$$-\frac{x}{y} = -\frac{\frac{1}{e}\sqrt{ab - b^2}}{\frac{b}{ae}\sqrt{a^2 - ab}} = -\frac{a}{b}\sqrt{\frac{b}{a}}.$$

The slope of the ellipse at the same point is

$$-\frac{b^2}{a^2}\times\frac{x}{y}=-\frac{b^2}{a^2}\cdot\frac{a}{b}\sqrt{\frac{b}{a}}=-\frac{b}{a}\sqrt{\frac{b}{a}}.$$

Whence, by [10], we have

$$\tan\phi=\pm\frac{-\frac{b}{a}\sqrt{\frac{b}{a}}+\frac{a}{b}\sqrt{\frac{b}{a}}}{1+\frac{b}{a}}=\frac{a-b}{\pm\sqrt{ab}}.$$

37. How would you draw a normal to an ellipse from any point in the minor axis?

Putting $x=0$ in the equation of the normal, we have

$$\text{Intercept on axis of } y=-\frac{c^2}{b^2}y_1.$$

Hence if on the axis of y we take OR so that

$$c^2:b^2::-\text{Intercept}:OR,$$

and through R draw a line parallel to the axis of x, its intersections with the curve will be the points to which normals can be drawn from the given point in the minor axis.

38. Find the equation of a chord bisected at a point (h, k).

The diameter bisecting this chord passes through (h, k). Substituting these values for x and y in the equation of a diameter, we obtain

$$k=-\frac{b^2h}{a^2m}, \text{ or } m=-\frac{b^2h}{a^2k}. \tag{1}$$

The line through (h, k), having the slope $-\frac{b^2h}{a^2k}$, is

$$y-k=-\frac{b^2h}{a^2k}(x-h), \text{ or } b^2hx+a^2ky=b^2h^2+a^2k^2.$$

39. Prove that the length of a line drawn from the centre to a tangent, and parallel to either focal radius of the point of contact, is equal to the semi-major axis.

OR and OH being respectively parallel to FP and $F'P$,

$$\angle OHR=\angle F'PR, \text{ and } \angle ORH=\angle FPH.$$

Hence $\angle ORH=\angle OHR$, and $OR=OH$.

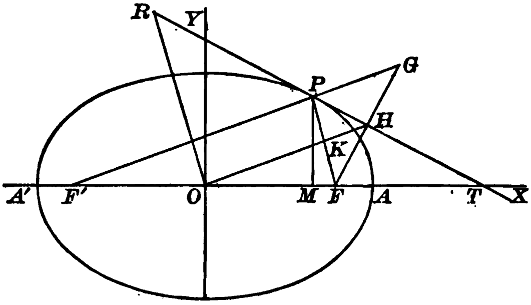

FIG. 82.

From the similar triangles ORT and FPT we have

$$FT : OT :: FP : OR. \tag{1}$$

But $\quad OT = \dfrac{a^2}{x_1},$

$$FT = OT - OF = \frac{a^2}{x_1} - ae = \frac{a(a - ex_1)}{x_1},$$

and $\quad FP = a - ex_1.$

Hence $\quad \dfrac{a(a - ex_1)}{x_1} : \dfrac{a^2}{x_1} :: a - ex_1 : OR,$ or $OR = a.$

40. A circle described on a focal radius will touch the auxiliary circle.

The circle described on FP (Fig. 82) will touch the auxiliary circle. Through O draw OH parallel to $F'G$. It will bisect FP and FG in K and H, respectively, and hence pass through the centre of the circle on FP. $OH = \frac{1}{2} F'G = a$. Angle $KHP = F'PR = KPH$; hence $KP = KH$, and

$$OK = OH - KH = a - KP.$$

Hence the circles are tangent, as the distance between their centres equals the difference of their radii.

41. Find the locus of the intersection of tangents drawn through the ends of conjugate diameters of an ellipse.

Let (x_1, y_1) be the end of one diameter; then, by § 144, the end of the other is $\left(\frac{-a}{b}y_1, \frac{b}{a}x_1\right)$. Hence the equations of the tangents are

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1, \tag{1}$$

and
$$\frac{-y_1x}{ab}+\frac{x_1y}{ab}=1. \tag{2}$$

Squaring both members of (1) and (2), adding, and remembering that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2, \text{ or } \frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1,$$

as the equation of the required locus, which is therefore an ellipse.

42. Find the locus of the middle point of the chord joining the ends of two conjugate diameters.

Let (x_1, y_1) be the end of one diameter; then $\left(\frac{-a}{b}y_1, \frac{b}{a}x_1\right)$ will be the end of the other. If (x, y) be the point mid way between them, by [2] we have

$$x=\frac{bx_1-ay_1}{2b},\quad y=\frac{ay_1+bx_1}{2a}. \tag{1}$$

Whence
$$x_1=\frac{bx+ay}{b},\quad y_1=\frac{ay-bx}{a}. \tag{2}$$

Substituting these value of x_1 and y_1 in

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1,$$

and simplifying, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}, \text{ or } \frac{x^2}{\frac{1}{2}a^2}+\frac{y^2}{\frac{1}{2}b^2}=1,$$

as the equation of the required locus, which is therefore an ellipse.

43. Find the locus of the vertex of a triangle whose base is the line joining the foci, and whose other sides are parallel to two conjugate diameters.

Let m and m' denote the slopes of the two sides; then their equations are

$$y=m(x-ae), \tag{1}$$

and
$$y=m'(x+ae). \tag{2}$$

But, by § 142,

$$mm' = \frac{-b^2}{a^2}, \text{ or } m' = \frac{-b^2}{a^2m}.$$

Substituting this value of m' in (2), we have

$$y = \frac{-b^2}{a^2m}(x + ae). \tag{3}$$

Equating the values of m obtained from (1) and (3), we obtain

$$a^2y^2 + b^2x^2 = b^2a^2e^2 = b^2c^2.$$

Hence the required locus is an ellipse whose semi-axes are c and $bc \div a$.

44. Show that $4x^2 + y^2 + 8x - 2y + 1 = 0$ represents an ellipse. Find its centre and axes.

The equation may be written in the form

$$4(x+1)^2 + (y-1)^2 = 4. \tag{1}$$

If we pass to parallel axes through $(-1, 1)$, the equation of the locus becomes

$$4x^2 + y^2 = 4, \text{ or } \frac{x^2}{1} + \frac{y^2}{4} = 1, \tag{2}$$

of which the locus is an ellipse whose transverse axis is 4, and lies upon the new axes of y, and whose conjugate axis is 2. The centre is $(-1, 1)$, referred to the old axes.

45. If A and B have like signs, show that the locus of

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

is in general an ellipse whose axes are parallel to the co-ordinate axes; and determine its semi-axes.

Since A and B have like signs, we may regard them as positive. The given equation may be written in the form

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + B\left(y^2 + \frac{E}{B}y + \frac{E^2}{4B^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4B},$$

or
$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 = -F + \frac{D^2}{4A} + \frac{E^2}{4B}. \tag{1}$$

Passing to parallel axes whose origin is $\left(\frac{-D}{2A}, \frac{-E}{2B}\right)$, and denoting the second member of (1) by k, we have

$$Ax^2 + By^2 = k, \tag{2}$$

in which A and B are positive.

If k is positive, the locus of (2) is an ellipse whose semi-axes are $\sqrt{\frac{k}{A}}$ and $\sqrt{\frac{k}{B}}$, and whose major axis lies on the axis of x, or the axis of y, according as $A <$ or $> B$; hence the locus of (1) is an ellipse whose axes are parallel to the axes of reference.

If $k = 0$, the locus of (2) is the point $(0, 0)$, or the locus of (1) is the point $\left(\frac{-D}{2A}, \frac{-E}{2B}\right)$.

If k is negative, (2) and hence (1) represents no real locus.

46. Find the locus of the centre of a circle which passes through the point $(0, 3)$ and touches internally the circle $x^2 + y^2 = 25$.

Let (x, y) be the centre whose locus is required; then the radius of the circle is the distance between $(0, 3)$ and (x, y), or $\sqrt{x^2 + (y-3)^2}$; and the distance between the centres of the two circles is $\sqrt{x^2 + y^2}$. But if the circles are tangent internally, the distance between their centres must be equal to the difference of their radii; hence we have

$$\sqrt{x^2 + y^2} = 5 - \sqrt{x^2 + (y-3)^2}.$$

Squaring and simplifying, and then squaring again, we obtain as the equation of the required locus

$$25x^2 + 16y^2 - 48y = 64. \qquad (1)$$

By No. 45 the locus of (1) is an ellipse.

Ex. 36. Page 174.

1. What is the equation of an hyperbola, if transverse axis $= 16$, conjugate axis $= 14$?

Here $a = 8$, $b = 7$; substituting these values in [40], we obtain

$$\frac{x^2}{64} - \frac{y^2}{49} = 1.$$

2. What is the equation of an hyperbola, if conjugate axes $= 12$, distance between foci $= 13$?

Here $b = 6$, $c = \frac{13}{2}$. Since $a^2 = c^2 - b^2$, we have $a^2 = \frac{25}{4}$.

Substituting these values of b and a in [40], we obtain

$$\frac{4x^2}{25} - \frac{y^2}{36} = 1.$$

3. What is the equation of an hyperbola, if distance between foci = twice the transverse axis?

Here $2c = 4a$, or $c = 2a$; hence $a^2 + b^2 = c^2 = 4a^2$; $\therefore\ b^2 = 3a^2$.

Substituting these values in [40], we have

$$\frac{x^2}{a^2} - \frac{y^2}{3a^2} = 1, \text{ or } 3x^2 - y^2 = 3a^2.$$

4. What is the equation of an hyperbola, if transverse axis = 8, one point is (10, 25)?

Here $a = 4$; hence the equation of the curve is

$$\frac{x^2}{16} - \frac{y^2}{b^2} = 1. \qquad (1)$$

Since (10, 25) is on (1), we have

$$\frac{100}{16} - \frac{625}{b^2} = 1, \text{ or } b^2 = \frac{2500}{21}.$$

Substituting this value of b^2 in (1), we have

$$\frac{x^2}{16} - \frac{21\,y^2}{2500} = 1, \text{ or } 625x^2 - 84y^2 = 10{,}000,$$

as the required equation.

5. What is the equation of an hyperbola, if distance between foci $= 2c$, eccentricity $= \sqrt{2}$?

Here $\frac{c}{a} = \sqrt{2}$, or $a^2 = \frac{1}{2}c^2$. $b^2 = c^2 - a^2 = c^2 - \frac{1}{2}c^2$. Substituting these values in [40], we have

$$2x^2 - 2y^2 = c^2.$$

6. Prove that the latus rectum of an hyperbola is equal to $\frac{2b^2}{a}$. Also $2a : 2b :: 2b :$ latus rectum.

Putting $x = c$ in the equation of the hyperbola [40], we have

$$y^2 = \frac{b^2}{a^2}(c^2 - a^2), \text{ or } y = \pm\frac{b^2}{a}.$$

Hence the latus rectum $= \frac{2b^2}{a}$ or $\frac{4b^2}{2a}$; whence $2a : 2b :: 2b :$ latus rectum.

7. The equation of an hyperbola is $9x^2 - 16y^2 = 144$; find the axes, distance between the foci, eccentricity, and latus rectum.

Putting the equation in the form of [40], we have

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

Hence $a = 4$ and $b = 3$. Hence $c^2 = a^2 + b^2 = 25$, or $2c = 10$, $e = \frac{5}{4}$, and $\frac{2b^2}{a}$, or latus rectum, is $\frac{9}{2}$.

8. Write the equation of the hyperbola conjugate to the hyperbola $9x^2 - 16y^2 = 144$, and find its axes, distance between its foci, and its latus rectum.

Writing the equation in the form of [40], we have

$$\frac{x^2}{16} - \frac{y^2}{9} = 1. \qquad (1)$$

Hence by (1) of § 158 we have as the equation of the conjugate hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = -1, \qquad (2)$$

of which the transverse axis is 6 and the conjugate 8. The distance between the foci is $2\sqrt{9+16}$, or 10, and the latus rectum is $\frac{2 \times 4^2}{3}$, or $\frac{32}{3}$.

In the formula, latus rectum $= \frac{2b^2}{a}$; it must be noted that b stands for the semi-conjugate diameter.

9. If the vertex of an hyperbola bisects the distance from the centre to the focus, find the ratio of its axes.

Here $a = \frac{1}{2}c$; hence b^2, or $c^2 - a^2$, $= \frac{3}{4}c^2$, or $b = \frac{1}{2}c\sqrt{3}$.
Therefore $a : b :: \frac{1}{2}c : \frac{1}{2}c\sqrt{3} :: 1 : \sqrt{3}$.

10. Prove that the point (x, y) is *without*, *on*, or *within* the hyperbola, according as $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ is *negative*, *zero*, or *positive*.

If $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, the point (x, y) must be on the hyperbola, by [40].

Through (x, y) draw a line parallel to the axis of x, cutting the curve in (x_1, y). Then

$$\frac{x_1^2}{a^2} - \frac{y^2}{b^2} - 1 = 0.$$

Now, if $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 < 0$, $x^2 < x_1^2$, and the point (x, y) lies between the curve and the axis of y; if $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 > 0$, $x^2 > x_1^2$, and the point (x, y) lies within the curve.

11. Find the eccentricity of an equilateral hyperbola.

Here $b = a$; $\therefore\ c^2 = 2a^2$, or $c = a\sqrt{2}$; hence $\frac{c}{a}$, or e, $= \sqrt{2}$.

12. Determine the points common to the hyperbola $25x^2 - 9y^2 = 225$, and the straight line $25x + 12y - 45 = 0$.

Regarding these equations as simultaneous, and solving for x and y, we find their common point to be $(5, -6\frac{2}{3})$. The straight line therefore is a tangent.

13. The asymptotes of an hyperbola are the diagonals of the rectangle on its axes.

The equations of these diagonals are $y = \pm\frac{b}{a}x$, which are also the equations of the asymptotes.

14. Find the foci and the asymptotes of the hyperbola

$$16x^2 - 9y^2 = 144.$$

Reducing the equation to the form of [40], we find $a = 3$, $b = 4$; hence $\sqrt{a^2 + b^2}$, or c, is 5, and the foci are $(\pm 5, 0)$. The equations of the asymptotes are

$$y = \pm\tfrac{4}{3}x.$$

15. The asymptotes of an equilateral hyperbola are perpendicular to each other. Hence the equilateral hyperbola is also called the *rectangular* hyperbola.

If $a = b$, the equations of the asymptotes are

$$y = \pm x, \tag{1}$$

which are perpendicular, by § 45, Cor. 2.

16. An hyperbola and its conjugate have the same asymptotes.

By § 159 the lines $y = \pm \frac{b}{a} x$ meet the conjugate hyperbola in two points at infinity, and hence by § 160 are asymptotes to it.

17. Find the length of a perpendicular dropped from the focus to an asymptote.

The distance from the focus $(c, 0)$ to the asymptote $ay - bx = 0$, is by [12],

$$\frac{-bc}{\sqrt{a^2 + b^2}}, \text{ or } b.$$

18. Brove that the squares of any two ordinates of the hyperbola are to each other as the products of the segments into which they divide the transverse axis externally.

Let (x_1, y_1) and (x_2, y_2) be any two points on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \text{ then we have}$$

$$y_1^2 = \frac{b^2}{a^2}(x_1^2 - a^2), \quad y_2^2 = \frac{b^2}{a^2}(x_2^2 - a^2).$$

Dividing and factoring, we have

$$y_1^2 : y_2^2 :: (x_1 - a)(x_1 + a) : (x_2 - a)(x_2 + a),$$

which proves the proposition.

Ex. 37. Page 176.

1. Find the equations of tangent and normal to the hyperbola $16x^2 - 9y^2 = 112$ at the point of contact (4, 4). Also find the lengths of the subtangent and the subnormal.

Here $a^2 = 7$, $b^2 = \frac{112}{9}$, $x_1 = 4$, $y_1 = 4$. Substituting these values in [43], [44], and the formulas of § 163, we obtain

$$\frac{4x}{7} - \frac{36y}{112} = 1, \text{ or } 16x - 9y = 28. \qquad (1)$$

$$y - 4 = -\frac{28}{\frac{4}{9} \times 112}(x - 4), \text{ or } 9x + 16y = 100. \qquad (2)$$

$$\text{Subt} = \frac{16 - 7}{4} = \frac{9}{4},$$

$$\text{Subn} = \frac{112}{9} \times \frac{4}{7} = \frac{64}{9}.$$

2. Show that in an equilateral hyperbola the subnormal is equal to the abscissa of the point of contact.

If $b = a$, $(b^2 \div a^2)x_1$, or subnormal, $= x_1$.

3. The equations of the tangent and the normal at a point of an equilateral hyperbola are $5x - 4y = 9$, $4x + 5y = 40$. What is the equation of the hyperbola, and what are the co-ordinates of the point of contact?

Regarding these equations as simultaneous we find that the common point of their loci is (5, 4), which is therefore the point of contact. The equation of any equilateral hyperbola is

$$x^2 - y^2 = a^2. \tag{1}$$

Since (5, 4) is on (1), we have $25 - 16 = 9 = a^2$. Hence the required equation is

$$x^2 - y^2 = 9.$$

4. For what points of an hyperbola is the subtangent equal to the subnormal?

If $$\frac{x_1^2 - a^2}{x_1} = \frac{b^2}{a^2}x_1, \quad x_1 = \frac{\pm a^2}{\sqrt{a^2 - b^2}}. \tag{1}$$

From (1) and the equation of the hyperbola, we obtain

$$y_1 = \frac{\pm b^2}{\sqrt{a^2 - b^2}}. \tag{2}$$

Hence there is one point in each quadrant, of which the co-ordinates are given in (1) and (2).

5. To draw a tangent and a normal to an hyperbola at a given point of the curve.

Draw the focal radii of the given point; the bisector of the included angle will be the required tangent by § 166. The line perpendicular to the tangent at the point of contact will be the normal.

6. If an ellipse and an hyperbola have the same foci, prove that the tangents to the two curves drawn at their points of intersection are perpendicular to each other.

The tangent to the hyperbola at a point of intersection will bisect the angle between the focal radii of the point (§ 116), and the tangent to the ellipse will bisect the supplement of this angle (§ 133). Hence the two tangents are perpendicular to each other.

7. Prove that the asymptotes of an hyperbola are the limiting positions of the tangents to the infinite branches.

The intercept of the tangent on the axis of x is $\frac{a^2}{x_1}$, whose limit is zero, as x_1 increases without limit. The slope of the tangent is

$$\frac{b^2x_1}{a^2y_1}, \text{ or } \pm\frac{b}{a\sqrt{1-\frac{a^2}{x_1^2}}},$$

whose limit is $\pm\frac{b}{a}$, as x_1 increases without limit.

Hence the asymptotes $y = \pm\frac{b}{a}x$ are the limiting positions of the tangents to the infinite branches, as the point of contact moves outward along a branch.

8. Prove that the length of a normal in an equilateral hyperbola is equal to the distance of the point of contact from the centre.

$$\text{Norm} = \sqrt{\text{subn}^2 + y_1^2} = \sqrt{\frac{b^4x_1^2}{a^4} + y_1^2}.$$

In an equilateral hyperbola $a = b$; hence,

$$\text{Norm} = \sqrt{x_1^2 + y_1^2},$$

or the distance of (x_1, y_1) from the centre $(0, 0)$.

9. Find the distance from the origin to the tangent through the end of the latus rectum of the equilateral hyperbola $x^2 - y^2 = a^2$.

One end of one latus rectum of $x^2 - y^2 = a^2$ is (ae, a), or $(a\sqrt{2}, a)$ since $e = \sqrt{2}$. The tangent at $(a\sqrt{2}, a)$ is

$$x\sqrt{2} - y = a.$$

whose distance from the origin is $\frac{a}{\sqrt{3}}$ or $\frac{1}{3}a\sqrt{3}$.

10. What condition must be satisfied in order that the straight line $\frac{x}{m} + \frac{y}{n} = 1$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

The equation of the tangent to the hyperbola in terms of its slope m_1 is

$$y = m_1x \pm \sqrt{a^2m_1^2 - b^2}. \tag{1}$$

Reducing the given equation to the form of (1), we have

$$y = -\frac{n}{m}x + n. \tag{2}$$

If (2) is a tangent we must have

$$m_1 = -\frac{n}{m},\quad n = \pm\sqrt{a^2m_1^2 - b^2}. \tag{3}$$

Substituting for m_1 its value, and squaring (3) becomes

$$\frac{a^2}{m^2} - \frac{b^2}{n^2} = 1,$$

which is therefore the condition required.

11. When is the director circle of an hyperbola imaginary?

When $a^2 - b^2$ is negative; that is, when the semi-transverse axis is greater than the semi-conjugate.

12. Find the locus of the foot of a perpendicular dropped from the focus of an hyperbola to a tangent.

The equation of the tangent is

$$y = mx + \sqrt{a^2m^2 - b^2}. \tag{1}$$

The equation of the perpendicular from the focus $(ae, 0)$ to (1), is

$$y = -\frac{1}{m}(x - ae). \tag{2}$$

From (1) $y - mx = \sqrt{a^2m^2 - b^2}$. (3)

From (2) $my + x = ae$. (4)

Squaring (3) and (4), adding, and dividing by $1 + m^2$, we obtain

$$x^2 + y^2 = a^2,$$

as the equation of the locus, which is a circle whose centre is $(0, 0)$ and whose radius is a.

Ex. 38. Page 177.

1. The ordinate through the focus of an hyperbola, produced, cuts the asymptotes in P and Q. Find PQ and the distances of P and Q from the centre.

The equation of PQ is $x = ae$. Combining this with the equation of the asymptotes $y = \pm\frac{b}{a}x$, we find $y = \pm be$; hence $PQ = 2be$.

The distance of (ae, be) from the origin is ce, or ae^2.

2. In the hyperbola $9x^2 - 16y^2 = 144$ what are the focal radii of the points whose common abscissa is 8? What other points have equal focal radii?

Since the point is on the right-hand branch

$$r = ex - a, \quad r' = ex + a. \qquad (1)$$

Here $a = 4$, $b = 3$; hence $c = 5$, and $e = \frac{5}{4}$. Substituting these values in (1), we obtain

$$r = \tfrac{5}{4}(8) - 4 = 6, \quad r' = \tfrac{5}{4}(8) + 4 = 14.$$

The points whose common abscissa is -8 have equal focal radii.

3. What relation exists between the *sum* of the focal radii of a point of an hyperbola and the abscissa of the point?

$$r + r' = ex - a + ex + a = 2ex.$$

4. Prove that in the equilateral hyperbola every ordinate is a mean proportional between the distances of its foot from the vertices of the curve. Hence find a method of constructing an equilateral hyperbola when the axes are given.

From the equation of the equilateral hyperbola, we have

$$y^2 = x^2 - a^2 = (x - a)(x + a),$$

or

$$x - a : y :: y : x + a,$$

which proves the proposition.

Hence to construct an equilateral hyperbola: at any point on the transverse axis produced erect a perpendicular, on this lay off a mean proportional between the distances of its foot from the vertices. The point thus determined will be a point on the hyperbola.

5. In the equilateral hyperbola the distance of a point from the centre is a mean proportional between its focal radii.

In the equilateral hyperbola $e = \sqrt{2}$; hence

$$r \times r' = (\sqrt{2}x - a)(\sqrt{2}x + a) = 2x^2 - a^2.$$

The distance from (x, y) to the centre is $\sqrt{x^2 + y^2}$, which, since (x, y) is on $x^2 - y^2 = a^2$, equals $\sqrt{x^2 + x^2 - a^2}$, or $\sqrt{2x^2 - a^2}$. Hence this distance equals the square root of the product of the focal radii, and is therefore a mean proportional between them.

6. In the equilateral hyperbola the bisectors of the angles formed by lines drawn from the vertices to any point of the curve are parallel to the asymptotes.

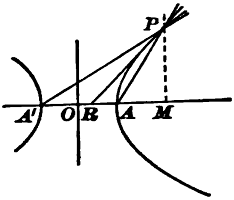

FIG. 83.

Let $P\ (x, y)$ be any point on the equilateral hyperbola

$$y^2 = x^2 - a^2. \tag{1}$$

Then

$$A'P^2 = A'M^2 + MP^2 = (x+a)^2 + x^2 - a^2,$$

and

$$AP^2 = AM^2 + MP^2 = (x-a)^2 + x^2 - a^2.$$

Or,

$$\left.\begin{aligned} A'P &= \sqrt{2x^2 + 2ax}, \\ AP &= \sqrt{2x^2 - 2ax}. \end{aligned}\right\} \tag{2}$$

Since PR bisects $A'PA$, we have

$$\frac{A'P}{AP} = \frac{A'R}{RA} = \frac{a+h}{a-h}, \tag{3}$$

if we denote OR by h. From (2) and (3), we have

$$\frac{a+h}{a-h} = \frac{\sqrt{2x^2+2ax}}{\sqrt{2x^2-2ax}} = \frac{\sqrt{x+a}}{\sqrt{x-a}}. \tag{4}$$

Taking (4) by division and composition, we have

$$\frac{h}{a} = \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}} = \frac{x - \sqrt{x^2 - a^2}}{a},$$

or

$$h = x - \sqrt{x^2 - a^2}. \tag{5}$$

But

$$\tan MRP = \frac{MP}{RM} = \frac{\sqrt{x^2 - a^2}}{x - h} = \frac{\sqrt{x^2 - a^2}}{\sqrt{x^2 - a^2}} = 1.$$

Hence PR is parallel to the asymptote, since the slope of each is unity.

7. If e, e' are the eccentricities of two conjugate hyperbolas,

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1.$$

Let

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2}; \text{ then } e'^2 = \frac{c^2}{b^2} = \frac{a^2 + b^2}{b^2}.$$

Hence

$$\frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

8. Through the positive vertex of an hyperbola a tangent is drawn. In what points does it cut the conjugate hyperbola?

The tangent through the vertex $(a, 0)$ is $x = a$. Substituting this value of x in the equation of the conjugate hyperbola, we find $y = \pm b\sqrt{2}$. Hence the points of intersection are $(a, b\sqrt{2})$ and $(a, -b\sqrt{2})$.

9. The sum of the reciprocals of two focal chords perpendicular to each other is constant.

From § 155 we have

$$r = ex - a. \tag{1}$$

Let θ denote the inclination of r to the axis of x;

then $x = r\cos\theta + ae.$

Substituting this value of x in (1), we have

$$r = er\cos\theta + ae^2 - a, \text{ or } r = \frac{a(e^2-1)}{1-e\cos\theta}. \tag{2}$$

Hence the length of the chord whose inclination is θ is

$$\frac{a(e^2-1)}{1-e\cos\theta} + \frac{a(e^2-1)}{1-e\cos(\theta+180^\circ)}, \text{ or } \frac{2a(e^2-1)}{1-e^2\cos^2\theta}, \tag{3}$$

and the length of the perpendicular chord is

$$\frac{a(e^2-1)}{1-e\cos(\theta+90^\circ)} + \frac{a(e^2-1)}{1-e\cos(\theta+270^\circ)},$$

or

$$\frac{2a(e^2-1)}{1-e^2\sin^2\theta}. \tag{4}$$

Adding the reciprocals of (3) and (4), we have

$$\frac{1-e^2\cos^2\theta}{2a(e^2-1)} + \frac{1-e^2\sin^2\theta}{2a(e^2-1)} = \frac{2-e^2}{2a(e^2-1)},$$

which proves the proposition.

10. Through the foot of the ordinate of a point in an equilateral hyperbola a tangent is drawn to the circle described upon the transverse axis as diameter. What relation exists between the lengths of this tangent and the ordinate of the point?

Let (x, y) be the point; then, since the radius of the circle is a, the length of the tangent is $\sqrt{x^2 - a^2}$. But the ordinate of the point $= \sqrt{x^2 - a^2}$; hence the ordinate and tangent are equal.

11. In an equilateral hyperbola find the equations of tangents drawn from the positive end of the conjugate axis.

The equation of any tangent to an equilateral hyperbola is

$$y = mx \pm a\sqrt{m^2 - 1}. \tag{1}$$

Since (1) passes through $(0, a)$, we have

$$a = \pm a\sqrt{m^2 - 1}, \text{ or } m = \pm \sqrt{2}.$$

Substituting this value of m in (1), we obtain

$$y = \pm \sqrt{2}\, x + a. \tag{2}$$

We use only the upper sign before a, since here the intercept is to be positive.

12. From what point in the conjugate axis of an hyperbola must tangents be drawn in order that they may be perpendicular to each other?

The equation of any tangent is

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{1}$$

Let (1) pass through $(0, y_1)$, any point on the conjugate axis; then we have from (1)

$$y_1 = \pm \sqrt{a^2m^2 - b^2}, \tag{2}$$

or

$$m^2 = \frac{y_1^2 + b^2}{a^2}. \tag{3}$$

That the two values of m in (3) may be negative reciprocals of each we must have $m^2 = 1$.

Substituting this value of m^2 in (3), and solving, we obtain

$$y_1 = \pm \sqrt{a^2 - b^2};$$

hence the points are

$$(0, \pm \sqrt{a^2 - b^2}).$$

13. What condition must be satisfied that a square may be constructed whose sides shall be parallel to the axes of an hyperbola and whose vertices shall lie in the curve?

The co-ordinates of the vertices of the square must evidently be equal; that is, if (x, y) be a vertex, $x^2 = y^2$. Hence, from the equation of the curve, we have

$$\frac{x^2}{a^2} - \frac{x^2}{b^2} = 1, \text{ or } x = \frac{\pm ab}{\sqrt{b^2 - a^2}}.$$

Now that the value of x may be real, we must have

$$b^2 - a^2 > 0, \text{ or } b^2 > a^2.$$

14. Find the equation of the chord of the hyperbola $16x^2 - 9y^2 = 144$ which is bisected at the point (12, 3).

Let m denote the slope of the chord, and (x_1, y_1) and (x_2, y_2) its extremities; then, from § 130, we have

$$m = \frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}. \tag{1}$$

But $a^2 = 9$, $b^2 = 16$, $x_1 + x_2 = 24$, and $y_1 + y_2 = 6$. Substituting these values in (1), we obtain

$$m = \tfrac{64}{9}.$$

The line through (12, 3), having slope $\tfrac{64}{9}$, is

$$y - 3 = \tfrac{64}{9}(x - 12), \text{ or } 64x - 9y - 741 = 0.$$

15. Find the equation of a tangent to the hyperbola $16x^2 - 9y^2 = 144$ parallel to the line $y = 4x - 3$.

The equation of a tangent is

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{1}$$

Here $m = 4$, $a^2 = 9$, $b^2 = 16$. Substituting these values in (1), we obtain

$$y = 4x \pm 8\sqrt{2}.$$

16. Find the product of the two perpendiculars let fall from any point of an hyperbola upon the asymptotes.

Let (x_1, y_1) be any point on the hyperbola; then its distances from the asymptotes $bx - ay = 0$ and $bx + ay = 0$ are

$$\frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}} \text{ and } \frac{bx_1 + ay_1}{\sqrt{a^2 + b^2}},$$

whose product is

$$\frac{b^2x_1^2 - a^2y_1^2}{a^2 + b^2}, \text{ or } \frac{a^2b^2}{a^2 + b^2}.$$

17. A chord of an hyperbola which touches the conjugate hyperbola is bisected at the point of contact.

If the x hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1}$$

the conjugate is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{2}$$

A tangent to (2) at (x_1, y_1) is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = -1. \tag{3}$$

Eliminating y between (1) and (3), we obtain

$$(a^2y_1^2 - x_1^2b^2)\,y^2 - 2\,a^2b^2y_1y = -\,a^2b^4 + x_1^2b^4,$$

or

$$y^2 - 2\,y_1y = -\,b^2 + x_1^2b^2 \div a^2. \tag{4}$$

Eliminating x between (1) and (3), we obtain

$$x^2 - 2\,x_1x = a^2 + y_1^2a^2 \div b^2. \tag{5}$$

Now the sum of the two values of y in (4) is $2\,y_1$, and the sum of the two values of x in (5) is $2\,x_1$; hence (x_1, y_1) is the point midway between the intersections of (1) and (3). That is, the part of the tangent to the conjugate hyperbola included between the branches of the x hyperbola is bisected at the point of contact.

Ex. 39. Page 188.

1. What is the polar of the point $(-9, 7)$ with respect to the hyperbola $7\,x^2 - 12\,y^2 = 112$?

The polar of the point $(-9, 7)$ with respect to $\frac{x^2}{\frac{112}{7}} - \frac{y^2}{\frac{112}{12}} = 1$ is

$$\frac{-9\,x}{\frac{112}{7}} - \frac{7\,y}{\frac{112}{12}} = 1, \text{ or } 9\,x + 12\,y + 16 = 0.$$

2. Find the equations of the directrices of an hyperbola.

By § 183 the equations are $x = \pm\frac{a}{e}$.

3. Find the angle formed by a focal chord and the line which joins its pole to the focus.

Since tangents at the extremities of a focal chord intersect in the directrix, the pole of every focal chord is some point $\left(\frac{a}{e}, k\right)$ in the directrix. The polar of $\left(\frac{a}{e}, k\right)$ is

$$\frac{ax}{ea^2} - \frac{ky}{b^2} = 1, \text{ or } y = \frac{b^2}{aek}\,x - \frac{b^2}{k}. \tag{1}$$

The line through $\left(\frac{a}{e}, k\right)$ and the focus $(ae, 0)$ is

$$y = \frac{k}{\frac{a}{e} - ae}(x - ae), \text{ or } y = \frac{aek}{-b^2}(x - ae). \qquad (2)$$

Since the slopes of lines (1) and (2) are negative reciprocals of each other, they are perpendicular to each other.

4. Find the pole of the line $Ax + By + C = 0$ with respect to an hyperbola.

Writing the given equation in the form of the polar of (h, k)

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1, \qquad (1)$$

we have
$$\frac{-(A \div C)a^2x}{a^2} - \frac{(B \div C)b^2y}{b^2} = 1. \qquad (2)$$

Comparing (1) and (2) we see that the pole (h, k) is in this case

$$\left(\frac{-Aa^2}{C}, \frac{Bb^2}{C}\right).$$

5. Find the polar of the right-hand vertex of an hyperbola with respect to the conjugate hyperbola.

The equation of the polar of the pole (h, k) with respect to the conjugate hyperbola is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = -1. \qquad (2)$$

If (h, k) is the right-hand vertex $(a, 0)$, (1) becomes

$$x + a = 0.$$

6. Find the distance from the centre of an hyperbola to the point where the directrix cuts the asymptote.

The directrix $x = \frac{a}{e}$, and the asymptote $y = \frac{b}{a}x$, intersect in the point $\left(\frac{a}{e}, \frac{b}{e}\right)$ whose distance from the origin is

$$\sqrt{\frac{a^2 + b^2}{e^2}} = \sqrt{\frac{c^2}{e^2}} = a.$$

7. If (x_1, y_1) and (x_2, y_2) are the ends of two conjugate diameters, then

$$\frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} = 0.$$

From the first of equations (2) in § 176, we have

$$y_2 = \frac{b^2x_1}{a^2y_1}x_2, \text{ or } \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} = 0.$$

8. The equation of a diameter in the hyperbola $25x^2 - 16y^2 = 400$ is $3y = x$. Find the equation of the conjugate diameter.

The equation of condition of conjugate diameters is

$$mm' = \frac{b^2}{a^2}.$$

Here $m = \frac{1}{3}$, $a^2 = 16$, $b^2 = 25$; hence $m' = \frac{75}{16}$, and the required equation is

$$y = \tfrac{75}{16}x, \text{ or } 75x - 16y = 0.$$

9. In the hyperbola $49x^2 - 4y^2 = 196$, find the equation of that chord which is bisected at the point (5, 3).

The equation of the diameter which bisects all chords whose slope is m, is

$$y = \frac{b^2}{a^2m}x. \tag{1}$$

Here $a^2 = 4$, $b^2 = 49$, and one point on the diameter is (5, 3). Substituting these values in (1), we obtain $m = \frac{245}{12}$.

The line through (5, 3), having slope $\frac{245}{12}$, is

$$y - 3 = \tfrac{245}{12}(x - 5), \text{ or } 245x - 12y - 1189 = 0.$$

10. Find the length of the semi-diameter conjugate to the diameter $y = 3x$ in the hyperbola $9x^2 - 4y^2 = 36$.

The equation of condition of conjugate diameters is

$$mm' = \frac{b^2}{a^2}. \tag{1}$$

Here $a^2 = 4$, $b^2 = 9$, and $m = 3$. Substituting these values in (1), we obtain $m' = \frac{3}{4}$; hence the equation of the conjugate diameter is

$$y = \tfrac{3}{4}x. \tag{2}$$

Regarding (2) and $9x^2 - 4y^2 = 36$ as simultaneous, we find $x^2 = \frac{16}{3}$ and $y^2 = 3$; hence $\sqrt{x^2 + y^2}$, or the distance of (x, y) from the centre is $\sqrt{\frac{16}{3} + 3}$, or $\frac{5}{3}\sqrt{3}$.

11. Two tangents to a hyperbola at right angles intersect on the circle $x^2 + y^2 = a^2 - b^2$.

This may be proved, as in the case of the ellipse § 136, or obtained by the principle of § 155.

12. Tangents at the extremities of any chord of an hyperbola intersect on the diameter which bisects that chord.

Take as the axis of x the diameter which bisects the chord, and as the axis of y the diameter parallel to the chord; then, if (x_1, y_1) be one extremity of the chord, $(x_1, -y_1)$ will be the other extremity, and the tangents will be

$$\frac{x_1 x}{a'^2} - \frac{y_1 y}{b'^2} = 1, \quad \frac{x_1 x}{a'^2} + \frac{y_1 y}{b'^2} = 1.$$

The intercept of each of these tangents on the axis of x is $\frac{a'^2}{x_1}$; hence they intersect on this axis, which is the diameter that bisects the chord.

13. Prove that PQ (Fig. 83) is parallel to one asymptote and bisected by the other.

The equation of PQ referred to the axes OP and OQ is

$$\frac{x}{OP} + \frac{y}{OQ} = 1, \text{ or } \frac{x}{a'} + \frac{y}{b'} = 1. \tag{1}$$

By § 177, the equation of the asymptote $C'S'$ is

$$\frac{x}{a'} + \frac{y}{b'} = 0. \tag{2}$$

Now (1) and (2) are evidently parallel.

Since $OPRQ$ is a parallelogram, PQ is bisected by OR.

14. An asymptote is its own conjugate diameter.

Since $mm' = \frac{b^2}{a^2}$, as m approaches $\frac{b}{a}$, m' approaches $\frac{b}{a}$ also, and when $m = \frac{b}{a}$, $m' = \frac{b}{a}$; hence, when one of two conjugate diameters coincides with an asymptote, the other coincides with the same asymptote.

15. The conjugate diameters of an equilateral hyperbola are equal.

From § 179, Cor. 3, we have

$$a'^2 - b'^2 = a^2 - b^2.$$

Hence, if $a = b,\ a' = b'.$

16. Having given two conjugate diameters in length and position, to find by construction the asymptotes and the axes.

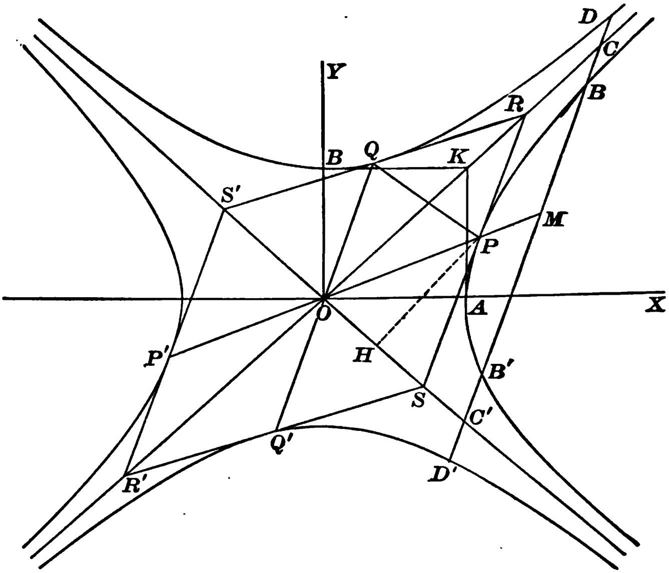

FIG. 84.

Let POP' and QOQ be the two given conjugate diameters. Construct the parallelogram $RS'R'S$ on these diameters; then the sides will be tangents, and the diagonals RR' and SS' will be the asymptotes by § 178. The axes will lie on the bisectors of the angles between the asymptotes.

On OR lay off OK so that $OR : OK :: OK : OS$ (§ 182, Cor. 3). From K draw perpendiculars to OX and OY. Denote their feet by A and B respectively; then OA and OB are the semi-axes required.

17. To draw a tangent to an hyperbola from a given point.

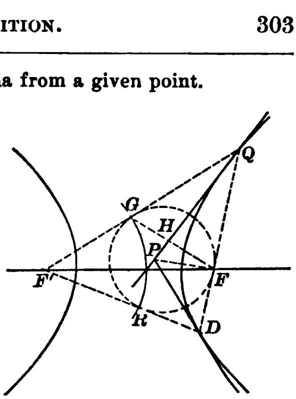

FIG. 85.

Let P be the given point.

With F' as a centre and $2a$ as a radius, draw the arc GR; with P as a centre and PF as a radius, draw circle GFR intersecting the first arc in G and R. Draw $F G$ and $F'R$ intersecting the curve in Q and D respectively. Join P with Q and D; then will PQ and PD be the tangents required.

For, since $F G = 2a$, $QG = QF$. Also $PG = PF$; hence the angles FQP and GQP are equal, and PQ is a tangent at Q. For like reason PD is a tangent at D.

18. Find the equation of a tangent at any point (x_1, y_1) of the hyperbola $4xy = a^2 + b^2$.

Let (x_1, y_1) and (x_2, y_2) be any two points on the curve. The equation of the secant through these points is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad (1)$$

Since (x_1, y_1) and (x_2, y_2) are on the curve, we have

$$4x_1y_1 = a^2 + b^2 = 4x_2y_2, \text{ or } y_2 = \frac{x_1y_1}{x_2},$$

which in (1) gives

$$y - y_1 = -\frac{y_1}{x_2}(x - x_1), \qquad (2)$$

which is the equation of the secant to the curve.

When (x_2, y_2) coincides with (x_1, y_1), (2) becomes

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1), \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2, \qquad (3)$$

which is the equation of the required tangent.

19. Find the equation of an hyperbola, taking as the axis of y

(i.) the tangent through the left-hand vertex;
(ii.) the tangent through the right-hand vertex.

(i.) To change the reference of the curve to parallel axes through $(-a, 0)$, we substitute $x - a$ for x, and y for y in [40]. We thus obtain as the required equation

$$\frac{(x-a)^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{2x}{a} = 0.$$

(ii.) Substituting $x + a$ for x, and y for y in [40], we obtain as the required equation

$$\frac{x^2}{a^3} - \frac{y^2}{b^2} + \frac{2x}{a} = 0.$$

20. Find the polar equation of an hyperbola, taking the right-hand focus as pole.

Here $\rho = ex - a,$ § 155

and $x = ae + \rho \cos \theta.$

Whence $\rho = ae^2 + \rho e \cos \theta - a$ or $\rho = \dfrac{a(e^2-1)}{1 - e\cos\theta}.$

21. Find the polar equation of an hyperbola, taking the centre as pole.

Substituting $\rho \cos \theta$ for x and $\rho \sin \theta$ for y in [40], we obtain

$$\frac{\rho^2 \cos^2 \theta}{a^2} - \frac{\rho^2 \sin^2 \theta}{b^2} = 1,$$

or

$$\rho^2 = \frac{a^2 b^2}{b^2 \cos^2 \theta - a^2 (1 - \cos^2 \theta)} = \frac{b^2}{e^2 \cos^2 \theta - 1}.$$

22. To find the centre of a given hyperbola.

Draw any two parallel chords: the line through their middle points is a diameter, the middle point of which is the centre of the given hyperbola.

23. The distance from a fixed point to a fixed straight line is 10. Find the locus of a point which moves so that its distance from the fixed point is always twice its distance from the fixed line.

I. *Solution.* From § 183, the locus is an hyperbola in which $e = 2$. To find its equation, take the fixed line OB, and a perpendicular to it through the fixed point F as the axes of y and x respectively. Let $P(x, y)$ be any position of the moving point; then

$$FP^2 = FM^2 + MP^2. \tag{1}$$

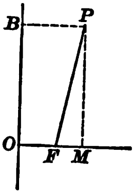

FIG. 86.

But $$FP = 2\,BP = 2 \times OM = 2\,x. \qquad (2)$$

$$MP = y,\ FM = x - OF = x - 10. \qquad (3)$$

Substituting these values in (1), we obtain

$$(2\,x)^2 = (x - 10)^2 + y^2,$$

or $$3\,x^2 - y^2 + 20\,x - 100 = 0. \qquad (4)$$

Equation (4) may be written in the form

$$3\left(x + \tfrac{10}{3}\right)^2 - y^2 = \tfrac{400}{3}. \qquad (5)$$

Passing to parallel axes through $(-\tfrac{10}{3}, 0)$, (5) becomes

$$9\,x^2 - 3\,y^2 = 400, \qquad (6)$$

which is the equation of the curve referred to its axes.

II. *Solution.* Here $e = 2$, $ae - \dfrac{a}{e} = 10$; hence $a^2 = \tfrac{400}{9}$.

From $b^2 = a^2e^2 - a^2$, we obtain $b^2 = \tfrac{400}{3}$.

Substituting these values in [40], we obtain

$$9\,x^2 - 3\,y^2 = 400.$$

24. Show that the locus of $x^2 - 4\,y^2 - 2\,x - 16\,y - 19 = 0$ is an hyperbola. Find its centre and axes.

The equation may be written in the form

$$(x - 1)^2 - 4\,(y + 2)^2 = 4. \qquad (1)$$

Passing to parallel axes whose origin is $(1, -2)$, the equation of the locus becomes

$$x^2 - 4\,y^2 = 4, \text{ or } \frac{x^2}{4} - \frac{y^2}{1} = 1,$$

the locus of which is an hyperbola in which $a = 2$ and $b = 1$.

25. If A and B have unlike signs, prove that the locus of

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

is in general an hyperbola whose axes are parallel to the co-ordinate axes; and determine its semi-axes.

Since A and B have unlike signs, we may regard A as positive and B negative. The given equation may be written in the form

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4\,A^2}\right) + B\left(y^2 + \frac{E}{B}y + \frac{E^2}{4\,B^2}\right) = -F + \frac{D^2}{4\,A} + \frac{E^2}{4\,B},$$

or $$A\left(x^2 + \frac{D}{2\,A}\right)^2 + B\left(y + \frac{E}{2\,B}\right)^2 = -F + \frac{D^2}{4\,A} + \frac{E^2}{4\,B}. \qquad (1)$$

Passing to parallel axes whose origin is $\left(-\frac{D}{4A}, -\frac{E}{2B}\right)$ and denoting the second member of (1) by k, we have

$$Ax^2 + By^2 = k, \tag{2}$$

in which A is positive and B negative. If k is positive, the locus of (2) is an hyperbola whose semi-axes are $\sqrt{\frac{k}{A}}$ and $\sqrt{\frac{k}{B}}$, the transverse axes lying on the axis of x. If k is negative, the transverse axis lies upon the axis of y. If $k = 0$, the locus is two intersecting straight lines through the origin.

26. Through the point $(-4, 7)$ a straight line is drawn to meet the axes of co-ordinates, and then revolved about this point. Find the locus of the point midway between the axes.

The equation of the line through $(-4, 7)$ is

$$y - 7 = m(x + 4). \tag{1}$$

For $x = 0$ in (1), $\quad y = 7 + 4m. \qquad (2)$

For $y = 0$ in (1), $\quad x = -\frac{7}{m} - 4. \qquad (3)$

Let (x', y') denote the point on (1) midway between the axes; then we have, from (1), (2), and (3),

$$y' - 7 = m(x' + 4), \tag{4}$$

$$y' = \tfrac{7}{2} + 2m, \tag{5}$$

$$x' = -\frac{7}{2m} - 2. \tag{6}$$

Finding the value of m in (5) or (6), substituting it in (4), and omitting the primes, we obtain as the equation of the locus

$$2xy - 7x + 4y = 0. \tag{1}$$

To reduce (1) to a known form, we pass to parallel axes whose origin is (h, k), and obtain

$$2xy + (2k - 7)x + (2h + 4)y + 2hk - 7h + 4k = 0. \tag{2}$$

If we choose h and k, so that $2h + 4 = 0$ and $2k - 7 = 0$, that is, if we take $h = -2$, $k = \frac{7}{2}$, the terms in x and y vanish, and the equation becomes $xy = -7$. From § 182, Cor. 1, we see that the locus of $xy = -7$ is an equilateral hyperbola, referred to its asymptotes, whose branches lie in the second and fourth quadrants.

27. A straight line has its ends in two fixed perpendicular lines, and forms with them a triangle of constant area a^2. Find the locus of the middle point of the line.

Let (x, y) be the middle point of the line; then $2xy$ will be the area of the triangle in any position of the line; hence the required equation is

$$2xy = a^2,$$

the locus of which is an equilateral hyperbola

28. The base a of a triangle is fixed in length and position, and the vertex so moves that one of the base angles is always double the other. Find the locus of the vertex.

Let OA be the fixed base a, AOB the smaller angle. Take OA as the axis of x, and O the origin. Let B be (x, y), and denote AOB by θ, and OAB by 2θ. Then

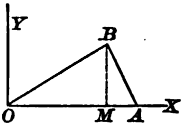

FIG. 87.

$$\tan\theta = \frac{y}{x}, \text{ and } \tan 2\theta = \frac{y}{a-x}.$$

But
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

$$\therefore \frac{y}{a-x} = \frac{2y}{x} \div \left(1 - \frac{y^2}{x^2}\right) = \frac{2xy}{x^2-y^2},$$

or
$$y(3x^2 - y^2 - 2ax) = 0.$$

Hence the locus consists of $y = 0$, or the axis of x, and the hyperbola $3x^2 - y^2 - 2ax = 0$.

Ex. 40. Page 206.

1. Determine the nature of the locus $3x^2 + 2y^2 - 2x + y - 1 = 0$, transform its equation, and construct it.

Here $A = 3,\ B = 2,\ C = 0,\ D = -2,\ E = 1,\ F = -1.$

Whence $\Sigma = 24,\ \Delta = 35.$

Hence the equation represents an ellipse or no real locus.

$$R = \Delta \div \Sigma = \tfrac{35}{24},\ h = \tfrac{1}{3},\ k = -\tfrac{1}{4}.$$

Therefore the equation of the locus referred to new parallel axes through the centre $(\frac{1}{3}, -\frac{1}{4})$ is (§ 188),

$$3x^2 + 2y^2 = \tfrac{35}{24}. \qquad (2)$$

From its form we see that (2) is the equation of an ellipse referred to its own axes.

Clearing (2) of fractions, we obtain

$$72x^2 + 48y^2 = 35, \text{ or } \frac{x^2}{\frac{35}{72}} + \frac{y^2}{\frac{35}{48}} = 1. \tag{3}$$

Whence $a = \sqrt{\frac{35}{48}}$, $b = \sqrt{\frac{35}{72}}$, and a lies on the axis of y.

To construct the equation, draw the axes O_1X_1, O_1Y_1; locate the centre $(\frac{1}{3}, -\frac{1}{4})$, and through this point draw the second set of axes O_2X_2, O_2Y_2. Lay off

$$O_2A = O_2A' = \sqrt{\tfrac{35}{48}} = 0.8,$$

and $$O_2B = O_2B' = \sqrt{\tfrac{35}{72}} = 0.7.$$

The ellipse having AA' and BB' as axes is the required locus.

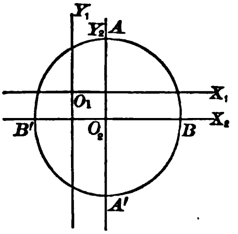

FIG. 88.

2. Determine the nature of the locus

$$3x^2 + 2xy + 3y^2 - 16y + 23 = 0,$$

transform its equation, and construct it.

Here $A = 3$, $B = 3$, $C = 2$, $D = 0$, $E = -16$, $F = 23$.

Whence $\Sigma = 32$, $\Delta = 32$.

Hence the equation represents an ellipse or no real locus.

$$R = \Delta \div \Sigma = 1, \quad h = -1, \quad k = +3.$$

Therefore the equation of the locus referred to new parallel axes through the centre $(-1, 3)$ is

$$3x^2 + 2xy + 3y^2 = 1.$$

To cause the term in xy to disappear, we have

$$\tan 2\theta = \frac{C}{A - B} = \frac{2}{0} = \infty.$$

Whence $2\theta = 90°$, or $\theta = 45°$.

$$P = \tfrac{1}{2}\left[A + B + \sqrt{(A - B)^2 + C^2}\right] = 4.$$

$$Q = \tfrac{1}{2}\left[A + B - \sqrt{(A - B)^2 + C^2}\right] = 2.$$

We use the upper signs in the values of P and Q, since C is positive. (§ 189, Cor. 3.)

Hence the equation of the ellipse referred to its own axis is

$$4x^2 + 2y^2 = 1, \text{ or } \frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{2}} = 1.$$

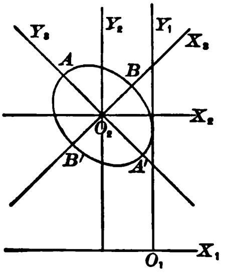

FIG. 89.

Whence $a = \sqrt{\frac{1}{2}}$, $b = \sqrt{\frac{1}{4}}$, and a lies on the axis of y. To construct the equation, draw the axes O_1X_1, O_1Y_1. Locate the centre $(-1, 3)$.

Through the centre O_2 draw the second set of axes O_2X_2, O_2Y_2.

Through O_2 draw the third set of axes O_2X_3, O_2Y_3 making $X_2O_2X_3$ equal to 45°.

Lay off $\quad O_2A = O_2A' = \sqrt{\frac{1}{2}} = 0.7,$

and $\quad O_2B = O_2B' = \sqrt{\frac{1}{4}} = 0.5.$

The ellipse having AA' and BB' as axes is the required locus.

3. Determine the nature of the locus

$$x^2 - 10xy + y^2 + x + y + 1 = 0, \tag{1}$$

transform its equation, and construct it.

Here $\quad A = 1,\ B = 1,\ C = -10,\ D = 1,\ E = 1,\ F = 1.$

Whence $\quad \Xi = -96,\ \Delta = +108.$

Hence the equation represents an hyperbola.

$$R = \Delta \div \Xi = -\tfrac{9}{8},\ h = \tfrac{1}{8},\ k = \tfrac{1}{8}.$$

Therefore the equation of the locus referred to new parallel axes through the centre $(\frac{1}{8}, \frac{1}{8})$ is (§ 188),

$$x^2 - 10xy + y^2 = -\tfrac{9}{8}. \tag{2}$$

To cause the term in xy to disappear we have

$$\tan 2\theta = \frac{C}{A-B} = \frac{-10}{0} = \infty.$$

Whence $\quad 2\theta = 90°$, or $\theta = 45°$.

$$P = \tfrac{1}{2}[A + B - \sqrt{(A-B)^2 + C^2}] = -4. \tag{3}$$

$$Q = \tfrac{1}{2}[A + B + \sqrt{(A-B)^2 + C^2}] = 6. \tag{4}$$

We use the lower signs in the values of P and Q, since C is negative.

Hence by § 189 the equation of the hyperbola referred to its own axes is

$$-4x^2 + 6y^2 = -\tfrac{9}{8}, \text{ or } 32x^2 - 48y^2 = 9,$$

from which we see that $a = \sqrt{\frac{9}{32}}$, $b = \sqrt{\frac{9}{48}}$ and a, or the transverse axis, lies on the axis of x.

To construct the equation, draw the axes O_1X_1, O_1Y_1, locate the centre $(\frac{1}{8}, \frac{1}{8})$, and through it draw the second set of axes O_2X_2, O_2Y_2. Through O_2 draw the third set of axes O_2X_3, O_2Y_3, making

$$X_2O_2X_3 = 45°;$$

then lay off

$$O_2A = O_2A' = \sqrt{\tfrac{9}{32}} = 0.5,$$

and $O_2B = O_2B' = \sqrt{\tfrac{9}{48}} = 0.4.$

The hyperbola having AA' and BB' as its transverse and conjugate axes respectively is the locus required.

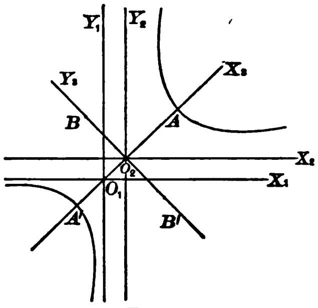

FIG. 90.

4. Determine the nature of the locus $x^2 + xy + y^2 + x + y - 5 = 0$, transform its equation, and construct it.

Here $\Delta = 16$, $\Sigma = 3$.

Therefore the locus is an ellipse, or there is no real locus.

$$R = \Delta \div \Sigma = \tfrac{16}{3}, \quad h = -\tfrac{1}{3}, \quad k = -\tfrac{1}{3}.$$

Hence the first transformed equation is

$$x^2 + xy + y^2 = \tfrac{16}{3}.$$

In the second transformation,

$$\theta = 45°, \quad P = \tfrac{3}{2}, \quad Q = \tfrac{1}{2}.$$

Hence the equation of the curve referred to its own axes is

$$9x^2 + 3y^2 = 32;$$

from which we see that $a = \sqrt{\tfrac{32}{3}}$, $b = \sqrt{\tfrac{32}{9}}$, and a, or the transverse axis, lies on the axis of y.

To construct the equation, draw the axes O_1X_1, O_1Y_1; locate the centre $(-\frac{1}{3}, -\frac{1}{3})$.

Through this centre O_2, draw the second set of axes, O_2X_2, O_2Y_2. Through O_2 draw the third set of axes, O_2X_3, O_2Y_3, making the angle $X_2O_2X_3$ equal to 45°.

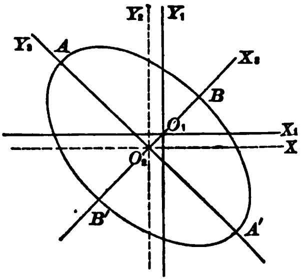

FIG. 91.

Lay off $O_2A = O_2A' = \sqrt{\tfrac{32}{3}} = 3.2$, and $O_2B = O_2B' = \sqrt{\tfrac{32}{9}} = 1.9$. The ellipse having AA' and BB' as axes will be the required locus.

5. Determine the nature of the locus $y^2 - x^2 - y = 0$, transform its equation, and construct it.

Here $\Sigma = -4$, $\Delta = -1$; hence the locus is an hyperbola.

$$R = \Delta \div \Sigma = \tfrac{1}{4}, \quad h = 0, \quad k = \tfrac{1}{2}.$$

Hence the first transformed equation is

$$y^2 - x^2 = \tfrac{1}{4}, \text{ or } \frac{x^2}{\frac{1}{4}} - \frac{y^2}{\frac{1}{4}} = -1,$$

from which we see that $a = \frac{1}{2}$, $b = \frac{1}{2}$, and the transverse axis lies on the axis of y.

To construct the equation, draw the axes O_1X_1, O_1Y_1, locate the centre $(0, \frac{1}{2})$, and through it draw the second set of axes O_2X_2, O_2Y_1.

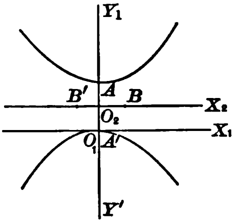

FIG. 92.

Lay off $O_2A = O_2A' = \frac{1}{2}$, and $O_2B = O_2B' = \frac{1}{2}$.

The hyperbola whose transverse axis is AA', and whose conjugate is BB', is the locus required.

6. Determine the nature of the locus $1 + 2x + 3y^2 = 0$, transform its equation, and construct it.

Here $\Sigma = 0$, and Δ is not zero; hence the locus is a parabola.

The given equation may be written in the form

$$y^2 = -\tfrac{2}{3}x - \tfrac{1}{3} = -\tfrac{2}{3}(x + \tfrac{1}{2}). \quad (1)$$

Passing to parallel axes whose origin is $(-\frac{1}{2}, 0)$, (1) becomes

$$y^2 = -\tfrac{2}{3}x, \quad (2)$$

of which the locus is a left-hand X-parabola whose latus rectum is $\frac{2}{3}$.

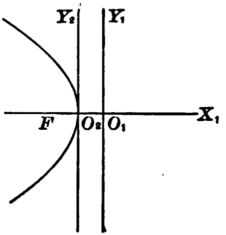

FIG. 93.

To construct the equation, draw the axes O_1X_1, O_1Y_1, locate the new origin $(-\frac{1}{2}, 0)$; then the second set of axes is O_2X_1, O_2Y_2 to which (2) refers the locus, which is now easily drawn.

7. Determine the nature of the locus $y^2 - 2xy + x^2 - 8x + 16 = 0$, transform its equation and construct it.

Here $\Sigma = 0$, Δ is not zero.

Therefore the locus is a parabola.

$$\theta = 45^\circ, \quad P = 0, \quad Q = 2, \quad U = -4\sqrt{2}, \quad V = 4\sqrt{2};$$

hence by revolving the axes through an angle of 45°, the equation becomes

$$y^2 - 2\sqrt{2}x + 2\sqrt{2}y + 8 = 0, \tag{1}$$

or

$$(y + \sqrt{2})^2 = 2\sqrt{2}\,(x - \tfrac{3}{2}\sqrt{2}). \tag{2}$$

Passing to parallel axes whose origin is $(\frac{3}{2}\sqrt{2}, -\sqrt{2})$, (2) becomes

$$y^2 = 2\sqrt{2}\,x, \tag{3}$$

of which the locus is a parabola, whose latus rectum is $2\sqrt{2}$.

To construct the equation draw the original axes O_1X_1, O_1Y_1; then draw the second set of axes O_1X_2, O_1Y_2, making

$$X_1O_1X_2 = 45°.$$

Locate the new origin $(\frac{3}{2}\sqrt{2}, -\sqrt{2})$, or $(2.1, -1.4)$, and through it draw the third set of axes O_3X_3, O_3Y_3, to which (3) refers the locus, which is now easily drawn.

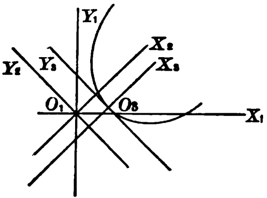

FIG. 94.

8. Determine the nature of the locus

$$x^2 - 2xy + y^2 - 6x - 6y + 9 = 0,$$

transform its equation, and construct it.

Here $\Sigma = 0$, and Δ is not zero.

Therefore the locus is a parabola,

$$\theta = 45°,\ P = 0,\ Q = 2,\ U = -6\sqrt{2},\ V = 0.$$

Hence by revolving the axes through an angle of 45°, the equation becomes $2y^2 - 6\sqrt{2}x + 9 = 0.$

$$\therefore\ y^2 = 3\sqrt{2}x - \tfrac{9}{2} = 3\sqrt{2}(x - \tfrac{3}{4}\sqrt{2}). \tag{1}$$

Passing to parallel axes whose origin is $(\frac{3}{4}\sqrt{2}, 0)$, equation (1) becomes $y^2 = 3x\sqrt{2}$, the locus of which is a right-hand X-parabola whose latus rectum is $3\sqrt{2}$.

To construct the equation, draw the original axes O_1X_1, O_1Y_1, then draw the second set of axes O_1X_2, O_1Y_2, making $X_1O_1X_2 = 45°$.

Locate the new origin $O_3(\frac{3}{4}\sqrt{2}, 0)$, or $(1.05, 0)$, and draw the third axis of Y parallel to O_1Y_2. Then O_3X_2, O_3Y_3 are the axes to which $y^2 = 3x\sqrt{2}$ refers the locus, which is now easily drawn.

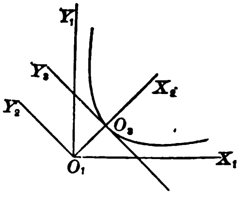

FIG. 95.

9. Determine the nature of the locus $y^2 - 2x - 8y + 10 = 0$, transform its equation, and construct it.

Here $\Sigma = 0$, Δ is not zero.

Therefore the locus is a parabola.

From the given equation we obtain

$$y^2 - 8y = 2x - 10; \tag{1}$$

hence $$y^2 - 8y + 16 = 2x + 6, \text{ or } (y-4)^2 = 2(x+3). \tag{2}$$

Passing to parallel axes whose origin is $(-3, 4)$, (2) becomes

$$y^2 = 2x, \tag{3}$$

the locus of which is a right-hand X-parabola, whose latus rectum, or parameter, is 2.

To construct the equation, draw the original axes O_1X_1', O_1Y_1; locate the new origin O_2 $(-3, 4,)$ and through it draw the second set of axes O_2X_2, O_2Y_2, to which (3) refers the locus, which is now easily drawn.

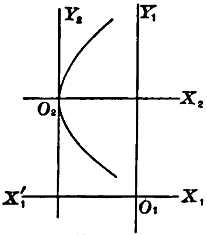

FIG. 96.

10. Determine the nature of the locus $4x^2 + 9y^2 + 8x + 36y + 4 = 0$, transform its equation, and construct it.

Here $\Sigma = 144$, and $\Delta = 5184$; hence the locus is an ellipse, or there is no real locus.

$$R = \Delta \div \Sigma = 36, \quad h = -1, \quad k = -2.$$

Hence the first transformed equation is

$$4x^2 + 9y^2 = 36,$$

from which we see that $a = 3$, and $b = 2$, and the transverse axis lies on the axis of x.

To construct the equation, draw the axes O_1X_1, O_1Y_1, locate the centre $(-1, -2)$, and draw through it the second set of axes O_2X_2, O_2Y_2.

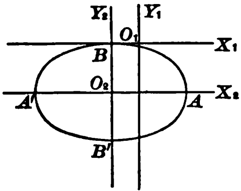

FIG. 97.

Lay off $O_2A = O_2A' = 3$, and $O_2B = O_2B' = 2$, and draw the ellipse having AA' and BB' as its axes.

11. Determine the nature of the locus $52x^2 + 72xy + 73y^2 = 0$, transform its equation, and construct it.

Here Σ is positive, and $\Delta = 0$; hence the locus is a point. This point is evidently the origin $(0, 0)$.

12. Determine the nature of the locus

$$9y^2 - 4x^2 - 8x + 18y + 41 = 0, \qquad (1)$$

transform its equation, and construct it.

Here Σ is negative, and Δ is not zero; hence the locus is an hyperbola.

The given equation may be written in the form

$$9(y^2 + 2y + 1) - 4(x^2 + 2x + 1) = -36,$$

or

$$4(x + 1)^2 - 9(y + 1)^2 = 36. \qquad (2)$$

Passing to parallel axes through $(-1, -1)$, (2) becomes

$$4x^2 - 9y^2 = 36, \text{ or } \frac{x^2}{9} - \frac{y^2}{4} = 1. \qquad (3)$$

The locus of (3) is an hyperbola whose semi-axes are 3 and 2 respectively, the transverse axis lying on the axis of x.

To construct the equation draw the axes O_1X_1, O_1Y_1; locate the new origin O_2, $(-1, -1)$, through which draw the new parallel axes.

Lay off $O_2A = O_2A' = 3$,
and $O_2B = O_2B' = 2$;
and draw the hyperbola whose transverse axis is AA' and whose conjugate is BB'.

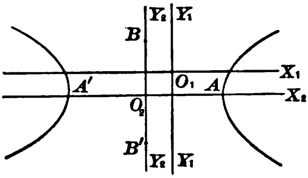

Fig. 98.

13. Determine the nature of the locus

$$y^2 - xy - 5x + 5y = 0, \qquad (1)$$

transform its equation, and construct it.

Here Σ is negative, and $\Delta = 0$; hence the locus is two intersecting right lines.

Writing (1) in the form

$$(y - x)(y + 5) = 0,$$

we see that its locus consists of the two right lines $y - x = 0$ and $y + 5 = 0$.

Ex. 41. Page 218.

1. Prove that the cissoid is the locus of the intersection of a tangent to the parabola $y^2 = -8rx$ with the perpendicular to it from the origin.

By § 112, the equation of the tangent to $y^2 = -8rx$ is

$$y = mx - \frac{2r}{m}, \tag{1}$$

and the perpendicular to it from (0, 0) is

$$y = -\frac{1}{m}x; \text{ therefore, } m = -\frac{x}{y}. \tag{2}$$

Eliminating m between (1) and (2), we obtain

$$y^2 = \frac{x^3}{2r - x}.$$

2. At the centre of any circle C (Fig. 99), erect CH perpendicular to the diameter OX; and on XO produced lay of $OA = OC = r$. Let LQR be a rectangular ruler of which the leg QR equals $2r$. If the ruler be moved so that the leg LQ slides along A, while the end R slides along CH, prove that the locus of P, the middle point of QR, is a cissoid.

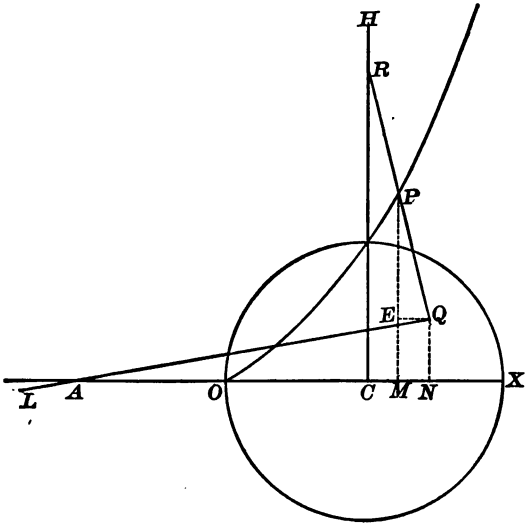

Fig. 99.

Let OX be the axis of x, O the origin, and AQR any position of the ruler; then $OM = x$, $MP = y$.

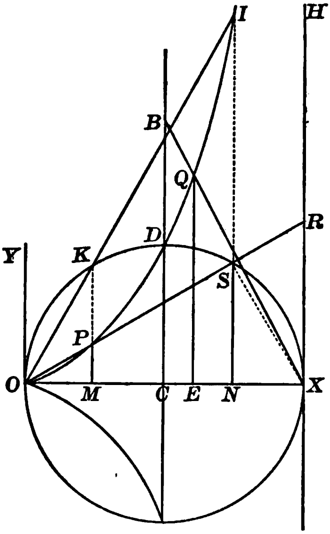

FIG. 100.

$$EQ = MN = CM = x - r, \quad AN = 2\,OM = 2x.$$

$$EP^2 = PQ^2 - EQ^2 = r^2 - (x - r)^2 = 2rx - x^2. \tag{1}$$

$$y = MP = NQ + EP. \tag{2}$$

$$NQ : AN :: EQ : EP,$$

or

$$NQ : 2x :: x - r : \sqrt{2rx - x^2}. \tag{3}$$

From (1), (2), and (3), we obtain

$$y = \frac{2x(x - r)}{\sqrt{2rx - x^2}} + \sqrt{2rx - x^2}, \text{ or } y^2 = \frac{x^3}{2r - x}.$$

This method of describing the cissoid by continuous motion was invented by Sir Isaac Newton.

3. In Fig. 100 prove that NS and ON are two mean proportionals between OM and NI; that is, prove

$$OM : NS : ON : NI.$$

$$NS^2 = NX \times ON = OM \times ON.$$

The right line OI will pass through K; hence

$$\angle OIN = \angle YOI = \tfrac{1}{2} \text{ arc } OK = \tfrac{1}{2} \text{ arc } SX$$
$$= \angle SOX;$$

therefore $NS : ON :: ON : NI.$

Hence $OM : NS : ON : NI.$

4. If in the lemniscate (Fig. 101), $OF' = OF = \frac{1}{2}a\sqrt{2}$, prove that $FP \times F'P$ is constant, P being any point on the curve; and hence that the lemniscate may be defined as the locus of a point, the product of whose distances from two fixed points is constant.

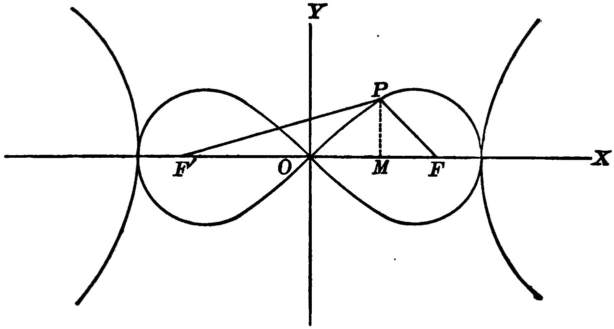

FIG. 101.

Draw PM parallel to OY; then

$$MF = \tfrac{1}{2}a\sqrt{2} - x, \quad MF' = \tfrac{1}{2}a\sqrt{2} + x.$$

$$\therefore \; FP = \sqrt{(\tfrac{1}{2}a\sqrt{2} - x)^2 + y^2}. \qquad (1)$$

$$F'P = \sqrt{(\tfrac{1}{2}a\sqrt{2} + x)^2 + y^2}. \qquad (2)$$

Multiplying (1) and (2), we have

$$FP \times F'P = \sqrt{(\tfrac{1}{2}a\sqrt{2} + x)^2 + y^2} \times \sqrt{(\tfrac{1}{2}a\sqrt{2} + x)^2 + y^2}$$
$$= \sqrt{(x^2 + y^2)^2 - a^2(x^2 - y^2) + \tfrac{1}{4}a^4} = \tfrac{1}{2}a^2, \text{ by [50]}.$$

That is, $FP \times F'P =$ the constant $\frac{1}{2}a^2$.

5. Construct the logarithmic curve $y = a^x$, or $x = \log_a y$. Prove that every logarithmic curve passes through the point (0, 1), and has the axis of x as an asymptote.

Assume $a = 10$; then $y = 10^x$.

$$\left.\begin{array}{l} \text{When } x = 0,\ 0.2,\ \ 0.4,\ \ 0.6,\ \ 0.8,\ \ 1, \\ \qquad\ \ y = 1,\ 1.58,\ 2.51,\ 3.98,\ 6.31,\ 10. \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l} \text{When } x = -0.1, -0.2, -0.4, -0.7, -1, -2, \\ \qquad\ \ y = 0.8,\ \ 0.6,\ \ 0.4,\ \ 0.2,\ \ 0.1,\ \ 0.01. \end{array}\right\} \quad (2)$$

Locating the points given in (1) and (2), and tracing the continuous curve RS through them, we have the locus that is called the logarithmic curve.

For $x = 0$ in $y = a^x$, $y = a^0 = 1$; hence whatever be the value of a, the locus of $y = a^x$ passes through the point (0, 1).

As y approaches zero in $x = \log_a y$, x approaches negative infinity; hence the infinite branch in the second quadrant has the axis of x as an asymptote whatever be the value of a.

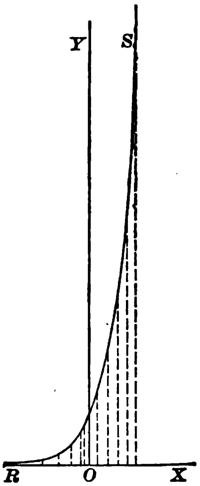

FIG. 102.

6. The *trochoid* is the curve traced by any point in the radius of a circle rolling on a right line. If r denote the radius of the circle, b the distance of the generating point from its centre, and θ denote the same angle as in § 202, show that the equations of the trochoid are

$$\left.\begin{array}{l} x = r\theta - b \sin\theta \\ y = r \quad - b \cos\theta \end{array}\right\}. \qquad (1)$$

When $b < r$, the trochoid is called the *prolate cycloid*; and when $b > r$, the *curtate cycloid*. When $b = r$, the curve is the *cycloid*.

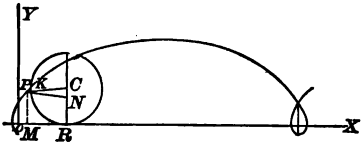

FIG. 103.

Let r denote the radius of the circle CRK which rolls on the line OX, and P be the point in the radius CK that traces the trochoid. Let

θ denote the angle RCP, and $b = CP$. Take the point O, in which K touched the line OX, as the origin and OX and OY as axes; then

$$x = OM = OR - PN = \text{arc } KR - CP \sin\theta,$$

or $$x = r\theta - b\sin\theta,$$

and $$y = MP = RC - NC = r - b\cos\theta.$$

When $b > r$, there is a loop in the curve which extends in part below the line. When $b = r$, the loop disappears in a cusp. When $b < r$ the cusp disappears.

Ex. 42. Page 229.

1. In what octants may (x, y, z) be, when x is positive? when x is negative? when y is positive? when y is negative? when z is positive? when z is negative?

1st, 4th, 5th, 8th; 2d, 3d, 6th, 7th; 1st, 2d, 5th, 6th; 3d, 4th, 7th, 8th; 1st, 2d, 3d, 4th; 5th, 6th, 7th, 8th.

2. In what octant is $(-2, 4, 6)$? $(2, 4, -3)$? $(-2, 4, -1)$? $(-2, -3, -1)$? $(-2, -3, 3)$? $(2, -3, 1)$? $(2, -1, -3)$? Construct each point.

2d; 5th; 6th; 7th; 3d; 4th; 8th;

3. In what line is $(a, 0, 0)$? $(0, 0, c)$? $(0, b, 0)$?

Axis of x; axis of z; axis of y.

4. In what plane is $(a, b, 0)$? $(a, 0, c)$? $(0, b, c)$?

Plane xy; plane xz; plane yz.

5. Find the length of the radius vector of $(3, 4, 5)$, $(2, -3, -1)$, $(7, -3, -5)$. Find the direction cosines of the radius vector of each point.

If ρ is the radius vector of (x, y, z), we have

$$\rho^2 = x^2 + y^2 + z^2. \qquad [53]$$

If (x, y, z) is $(3, 4, 5)$,

$$\rho = \sqrt{9 + 16 + 25} = 5\sqrt{2}.$$

If (x, y, z) is $(2, -3, -1)$,

$$\rho = \sqrt{4 + 9 + 1} = \sqrt{14}.$$

If (x, y, z) is $(7, -3, -5)$,

$$\rho = \sqrt{49 + 9 + 25} = \sqrt{83}.$$

By [54], $\cos\alpha = x \div \rho$, $\cos\beta = y \div \rho$, $\cos\gamma = z \div \rho$.

If (x, y, z) is $(3, 4, 5)$, $\rho = 5\sqrt{2}$; hence $\cos\alpha = 3 \div 5\sqrt{2} = 0.3\sqrt{2}$, $\cos\beta = 4 \div 5\sqrt{2} = 0.4\sqrt{2}$, $\cos\gamma = 5 \div 5\sqrt{2} = 0.5\sqrt{2}$.

If (x, y, z) is $(2, -3, -1)$, $\rho = \sqrt{14}$; hence $\cos\alpha = 2 \div \sqrt{14} = \frac{1}{7}\sqrt{14}$, $\cos\beta = -3 \div \sqrt{14} = -\frac{3}{14}\sqrt{14}$, $\cos\gamma = -1 \div \sqrt{14} = -\frac{1}{14}\sqrt{14}$.

If (x, y, z) is $(7, -3, -5)$, $\rho = \sqrt{83}$; hence $\cos\alpha = 7 \div \sqrt{83} = \frac{7}{83}\sqrt{83}$, $\cos\beta = -3 \div \sqrt{83} = -\frac{3}{83}\sqrt{83}$, $\cos\gamma = -5 \div \sqrt{83} = -\frac{5}{83}\sqrt{83}$.

6. The direction cosines of a line are proportional to 1, 2, 3; find their values. What is the direction of the line?

Dividing each of the three real quantities 1, 2, 3 by $\sqrt{1+4+9}$, or $\sqrt{14}$, we obtain as the direction cosines of the line $\frac{1}{14}\sqrt{14}$, $\frac{1}{7}\sqrt{14}$, $\frac{3}{14}\sqrt{14}$. The line is the radius vector of the point (1, 2, 3), or it is a line parallel to this radius vector.

7. What is the direction of the line whose direction cosines are proportional to A, B, C? What are the values of its direction cosines?

The line has the direction of the radius vector of the point (A, B, C). By § 213, Cor., the direction cosines of the line are

$$\frac{A}{\sqrt{A^2+B^2+C^2}},\quad \frac{B}{\sqrt{A^2+B^2+C^2}},\quad \frac{C}{\sqrt{A^2+B^2+C^2}}.$$

8. Two direction angles of a line are 60° and 45°, what is the third? If two are 60° and 30°, what is the third? If two are 135° and 60°, what is the third?

If α, β, γ are the direction cosines of a line, we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad [55]$$

If $\alpha = 60°$ and $\beta = 45°$, $\cos\alpha = \frac{1}{2}$, and $\cos\beta = \frac{1}{2}\sqrt{2}$. Substituting these values in [55], we obtain

$$\tfrac{1}{4} + \tfrac{1}{2} + \cos^2\gamma = 1, \text{ or } \cos\gamma = \pm\tfrac{1}{2};$$

hence $\gamma = 60°$, or $120°$.

If $\alpha = 60°$ and $\beta = 30°$,

$$\cos\alpha = \tfrac{1}{2} \text{ and } \cos\beta = \tfrac{1}{2}\sqrt{3};$$

hence $\frac{1}{4} + \frac{3}{4} + \cos^2\gamma = 1$, or $\cos\gamma = 0$;

hence $\gamma = 90°$,

and the line lies in the plane xy.

If $\alpha = 135^\circ$ and $\beta = 60^\circ$,

$$\cos\alpha = -\tfrac{1}{2}\sqrt{2} \text{ and } \cos\beta = \tfrac{1}{2};$$

hence $\tfrac{1}{2} + \tfrac{1}{4} + \cos^2\gamma = 1$, or $\cos\gamma = \pm\tfrac{1}{2}$,

and $\gamma = 60^\circ$ or 120°.

Ex. 43. Page 235.

1. Find the distance between the points (1, 2, 3), (2, 3, 4); (2, 3, 4), (3, 4, 5); (1, 2, 3), (3, 4, 5).

If D denote the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad [58]$$

If (x_1, y_1, z_1) is (1, 2, 3), and (x_2, y_2, z_2) is (2, 3, 4),

$$D = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

If (x_1, y_1, z_1) is (2, 3, 4), and (x_2, y_2, z_2) is (3, 4, 5),

$$D = \sqrt{3}.$$

If (x_1, y_1, z_1) is (1, 2, 3), and (x_2, y_2, z_2) is (3, 4, 5),

$$D = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}.$$

2. Prove that the triangle formed by joining the three points (1, 2, 3), (2, 3, 1), (3, 1, 2) is equilateral.

First, let (x_1, y_1, z_1) be (1, 2, 3), and (x_2, y_2, z_2) be (2, 3, 1); then

$$D = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

Second, let (x_1, y_1, z_1) be (2, 3, 1), and (x_2, y_2, z_2) be (3, 1, 2); then

$$D = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

Third, let (x_1, y_1, z_1) be (3, 1, 2), and (x_2, y_2, z_2) be (1, 2, 3); then

$$D = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}.$$

Hence each side of the triangle is $\sqrt{6}$, and it is therefore equilateral.

3. The lengths of the projections of a line on the three co-ordinate axes are 3, 4, 5, respectively; find the length of the line.

Let PP' be the line; then, by § 218, Cor. 1, we have

$$PP'^2 = 3^2 + 4^2 + 5^2 = 50, \text{ or } PP = 5\sqrt{2}.$$

4. Find the direction cosines of the radius vector of the point $(-3, -4, 5)$.

Here, $\rho = \sqrt{(-3)^2 + (-4)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$;

hence, by § 213, Cor., we have

$$\cos\alpha = -\tfrac{3}{10}\sqrt{2},\ \cos\beta = -\tfrac{2}{5}\sqrt{2},\ \cos\gamma = \tfrac{1}{2}\sqrt{2}.$$

5. What lines have direction cosines proportional to $3, -2, -5$? Find the values of these direction cosines.

By § 213, Cor., we see that the direction cosines of the radius vector of the point $(3, -2, -5)$ are proportional to $3, -2, -5$. Hence all lines parallel to this radius vector have direction cosines proportional to $3, -2, -5$. By the same Cor. the values of these direction cosines are $\frac{3}{\sqrt{38}}$, $\frac{-2}{\sqrt{38}}$ and $\frac{-5}{\sqrt{38}}$.

6. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3, and 2, 3, 6 respectively.

If α, β, γ denote the direction angles of the first line, α', β', γ' the direction angles of the second, and θ the angle between the lines; then we have

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \qquad [57]$$

Here $\cos\alpha = \frac{1}{\sqrt{14}}$; $\cos\beta = \frac{2}{\sqrt{14}}$; $\cos\gamma = \frac{3}{\sqrt{14}}$.

$\cos\alpha' = \frac{2}{7}$; $\cos\beta' = \frac{3}{7}$; $\cos\gamma' = \frac{6}{7}$.

Substituting these values in [57], we obtain

$$\cos\theta = \tfrac{13}{49}\sqrt{14}.$$

Hence $\theta = \cos^{-1}\tfrac{13}{49}\sqrt{14}$.

7. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3, and 5, -4, 1, respectively.

Let the direction angles of the lines be denoted as in No. 6; also the angle included between the lines.

Here $\cos\alpha = \frac{1}{\sqrt{14}}$; $\cos\beta = \frac{2}{\sqrt{14}}$; $\cos\gamma = \frac{3}{\sqrt{14}}$.

$\cos\alpha' = \frac{5}{\sqrt{42}}$; $\cos\beta' = \frac{-4}{\sqrt{42}}$; $\cos\gamma' = \frac{1}{\sqrt{42}}$.

Substituting in [57] of No. 6, we have

$$\cos\theta = 0.$$

Hence $\theta = 90^\circ$.

8. Find the polar co-ordinates of the point $(\sqrt{3}, 1, 2\sqrt{3})$.

By § 219, Cor., we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3 + 1 + 12} = 4;$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z} = \frac{\sqrt{3 + 1}}{2\sqrt{3}} = \tfrac{1}{3}\sqrt{3}, \text{ or } \theta = 30^\circ;$$

and

$$\tan\phi = \frac{y}{x} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}, \text{ or } \phi = 30^\circ.$$

9. Find the rectangular co-ordinates of the point $(4, \frac{1}{6}\pi, \frac{1}{3}\pi)$.

By § 219, Cor., we have

$$x = \rho\sin\theta\cos\phi = 4\sin 30^\circ\cos 60^\circ = 1,$$

$$y = \rho\sin\theta\sin\phi = 4\sin 30^\circ\sin 60^\circ = \sqrt{3},$$

$$z = \rho\cos\theta = 4\cos 30^\circ = 2\sqrt{3}.$$

10. If (x, y, z) bisect the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2), prove that $x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$, $z = \frac{1}{2}(z_1 + z_2)$.

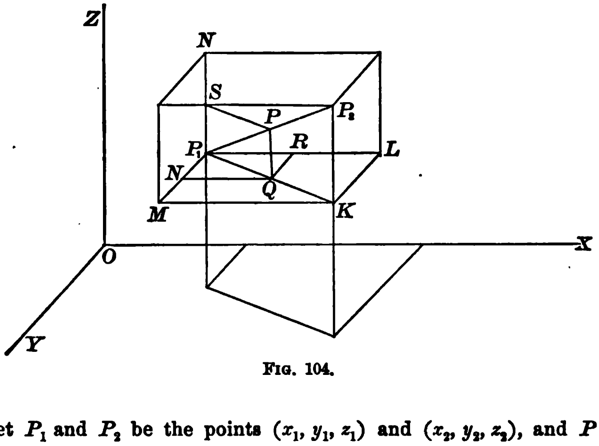

FIG. 104.

Let P_1 and P_2 be the points (x_1, y_1, z_1) and (x_2, y_2, z_2), and P the point (x, y, z) bisecting the line P_1P_2. Through P_1 and P_2 pass planes parallel to the co-ordinate planes thus forming the rectangular parallelopiped P_2P_1MKL. Through P pass planes parallel to the co-ordinate planes cutting P_1L, P_1M, P_1N in R, N, S, respectively; then

$$P_1R = RL, \quad P_1N = NM, \quad P_1S = SN.$$

But
$$P_1R = x - x_1, \quad RL = x_2 - x, \quad P_1N = y - y_1,$$
$$NM = y_2 - y, \quad P_1S = z - z_1, \quad SN = z_2 - z.$$

Hence, by substitution,

$$x - x_1 = x_2 - x, \text{ or } x = \tfrac{1}{2}(x_1 + x_2);$$
$$y - y_1 = y_2 - y, \text{ or } y = \tfrac{1}{2}(y_1 + y_2);$$
$$z - z_1 = z_2 - z, \text{ or } z = \tfrac{1}{2}(z_1 + z_2).$$

11. If (x, y, z) divide the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2), in the ratio $m_1 : m_2$, prove that

$$x = \frac{m_2x_1 + m_1x_2}{m_2 + m_1}, \quad y = \frac{m_2y_1 + m_1y_2}{m_2 + m_1}, \quad z = \frac{m_2z_1 + m_1z_2}{m_2 + m_1}.$$

Using the same notation and construction as in No. 10, suppose $P(x, y, z)$ divides the line P_1P_2 into segments in the ratio $m_1 : m_2$; then

$$x - x_1 : x_2 - x :: m_1 : m_2, \text{ or } x = \frac{m_2x_1 + m_1x_2}{m_2 + m_1};$$
$$y - y_1 : y_2 - y :: m_1 : m_2, \text{ or } y = \frac{m_2y_1 + m_1y_2}{m_2 + m_1};$$
$$z - z_1 : z_2 - y :: m_1 : m_2, \text{ or } z = \frac{m_2z_1 + m_1z_2}{m_2 + m_1}.$$

12. Find the co-ordinates of the point that divides the line joining $(3, -2, 4)$ and $(1, 3, -2)$ in the ratio $1 : 3$.

Here (x_1, y_1, z_1) is $(3, -2, 4)$, (x_2, y_2, z_2) is $(1, 3, -2)$ and $m_1 : m_2$ is $1 : 3$. Substituting these values in the formulas in No. 11, we obtain

$$x = \tfrac{5}{2}, \quad y = -\tfrac{3}{4}, \quad z = \tfrac{5}{2}.$$

13. Find the point that divides the line joining $(-2, -3, -1)$ and $(-5, -2, 4)$ in the ratio $5 : 2$.

Here (x_1, y_1, z_1) is $(-2, -3, -1)$, (x_2, y_2, z_2) is $(-5, -2, 4)$, and $m_1 : m_2$ is $5 : 2$. Substituting these values in the formulas in No. 11, we obtain

$$x = -\tfrac{29}{7}, \quad y = -\tfrac{16}{7}, \quad z = \tfrac{18}{7}.$$

Ex. 44. Page 240.

1. To which co-ordinate plane is $3y-4z=2$ perpendicular? $x-8z-7=0$? $x-2y=2$? $x=mz+p$? $y=nx+q$? What is the locus of $z=5$? $y=-7$? $y=4$? $z=-2$? $x=0$? $y=0$? $z=0$?

By § 222, Cor. 3, the plane yz; the plane zx; the plane xy; the plane zx; the plane xy; a plane perpendicular to the axis of z, 5 above the origin; a plane perpendicular to the axis of y, 7 back from the origin; a plane perpendicular to the axis of y, 4 in front of the origin; a plane perpendicular to the axis of z, 2 below the origin; the plane yz; the plane zx; the plane xy.

2. Reduce to the normal form

$$3x-2y+z=2;\ 5x-4y+z=4.$$

What is the distance of each of these planes from the origin? What are the direction cosines of the perpendiculars to each? Which of the eight octants does each truncate?

By § 222, Cor. 2, to reduce $3x-2y+z=2$ to the normal form we divide by $\sqrt{3^2+(-2)^2+1^2}$ or $\sqrt{14}$, and obtain

$$\frac{3}{\sqrt{14}}x-\frac{2}{\sqrt{14}}y+\frac{1}{\sqrt{14}}z=\frac{2}{\sqrt{14}}=\tfrac{2}{14}\sqrt{14}. \tag{1}$$

To reduce $5x-4y+z=4$ to the normal form we divide by

$$\sqrt{5^2+(-4)^2+1^2} \text{ or } \sqrt{42},$$

and obtain

$$\frac{5}{\sqrt{42}}x-\frac{4}{\sqrt{42}}y+\frac{1}{\sqrt{42}}z=\frac{4}{\sqrt{42}}=\tfrac{2}{21}\sqrt{42}. \tag{2}$$

The distance of (1) from the origin is $\frac{2}{14}\sqrt{14}$, while the distance of (2) from the origin is $\frac{2}{21}\sqrt{42}$ (§ 222, Cor. 1). The direction cosines of perpendiculars to (1) are $\frac{3}{\sqrt{14}}$, $\frac{-2}{\sqrt{14}}$, and $\frac{1}{\sqrt{14}}$ respectively; those of perpendiculars to (2) are $\frac{5}{\sqrt{42}}$, $\frac{-4}{\sqrt{42}}$, and $\frac{1}{\sqrt{42}}$ (§ 222, Cor. 1).

Since the point $(3, -2, 1)$ is in the fourth octant (1) truncates the fourth octant (§ 222, Cor. 1). Since $(5, -4, 1)$ is in the fourth octant (2) truncates the fourth octant.

3. Find the intercepts on the axes of $3x-2y+4z-12=0$; of $6x-4y-3z+24=0$; $5x+7y+5z+35=0$. Which of the eight octants does each truncate?

Reduce each equation to the symmetrical form.

Reducing each of these equations to the symmetrical form, we obtain

$$\frac{x}{4}+\frac{y}{-6}+\frac{z}{3}=1, \tag{1}$$

$$\frac{x}{-4}+\frac{y}{6}+\frac{z}{8}=1, \tag{2}$$

$$\frac{x}{-7}+\frac{y}{-5}+\frac{z}{-7}=1. \tag{3}$$

The intercepts on the axes of (1) are 4, -6, and 3, and (1) truncates the fourth octant.

The intercept on the axes of (2) are -4, 6, and 8, and (2) truncates the second octant.

The intercepts of (3) are -7, -5, and -7, and (3) truncates the seventh octant.

4. What is the equation of the plane at the distance 7 from the origin, and perpendicular to a line whose direction cosines are proportional to 2, -3, and $\sqrt{3}$?

If the direction cosines of the perpendiculars to the plane are proportional to 2, -3, and $\sqrt{3}$, the direction cosines themselves are $\frac{1}{2}$, $-\frac{3}{4}$, and $\frac{1}{4}\sqrt{3}$ (§ 222, Cor. 1); hence the required equation by [59] is

$$\tfrac{1}{2}x-\tfrac{3}{4}y+\tfrac{1}{4}\sqrt{3}z=7, \text{ or } 2x-3y+\sqrt{3}z=28.$$

5. What is the equation of the plane whose intercepts on the axes are respectively 4, -3, -7? -1, -2, -5? $\frac{1}{2}$, $-\frac{2}{3}$, $\frac{1}{6}$?

By § 223, we have as the required equations

$$\frac{x}{4}-\frac{y}{3}-\frac{z}{7}=1, \quad \frac{x}{1}+\frac{y}{2}+\frac{z}{5}=-1,$$

and $$2x-\tfrac{3}{2}y+6z=1.$$

6. Find the equation of the plane passing through the points (1, 2, 3), (0, 4, -1), and (1, -1, 0).

If the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ passes through each of the given points, we have

$$\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1, \tag{1}$$

$$\frac{4}{b}-\frac{1}{c}=1, \tag{2}$$

$$\frac{1}{a}-\frac{1}{b}=1. \tag{3}$$

Subtracting (3) from (1), we have

$$\frac{1}{b}+\frac{1}{c}=0. \tag{4}$$

(4) + (2) gives $\frac{5}{b}=1$, or $b=5$.

$\therefore\ a=\frac{1}{6}$, $c=-5$. Substituting these values in [60], we obtain

$$\frac{x}{\frac{1}{6}}+\frac{y}{5}+\frac{c}{-5}=1, \text{ or } 6x+y-z=5.$$

7. Find the angle between the two planes

$$2x+z-y=3,$$
$$z+x+2y=5.$$

If θ denotes the angle between the two planes

$$Ax+By+Cz=D,\ A'x+B'y+C'z=D';$$

then

$$\cos\theta=\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}. \tag{[61]}$$

Here $A=2$, $B=-1$, $C=1$, $A'=1$, $B'=2$, $C'=1$. Substituting these values in [61], we have

$$\cos\theta=\frac{2-2+1}{\sqrt{4+1+1}\sqrt{1+1+4}}=\frac{1}{6}, \text{ or } \theta=\cos^{-1}\frac{1}{6}.$$

8. Find the angle between the planes

$$3z+5x-7y=-1,\ 3z-2x-y=0.$$

Here $A=-5$, $B=7$, $C=-3$, $A'=2$, $B'=1$, $C'=-3$. Substituting these values in [61], we have

$$\cos\theta=\frac{-10+7+9}{\sqrt{9+25+49}\sqrt{9+4+1}}=\frac{6}{\sqrt{1162}}.$$

$$\therefore \log\cos\theta=\log 6-\tfrac{1}{2}\log 1162=9.24555-10.$$

$$\therefore \theta=79^\circ\ 52'.$$

9. Find the angle which the plane $Ax + By + Cz = D$ makes with each of the co-ordinate planes.

From § 221, Cor. 3, and § 222, we have

$\alpha = \cos^{-1}\dfrac{A}{\sqrt{A^2+B^2+C^2}}$ = the angle which $Ax + By + Cz = D$ makes with the plane yz.

$\beta = \cos^{-1}\dfrac{B}{\sqrt{A^2+B^2+C^2}}$ = the angle which the plane makes with the plane zx.

$\gamma = \cos^{-1}\dfrac{C}{\sqrt{A^2+B^2+C^2}}$ = the angle which the plane makes with the plane xy.

Or we could, by [61], find the angle between the given plane and the planes $x = 0$, $y = 0$, and $z = 0$, respectively.

10. Find the distance from the point $(2, -3, 0)$ to the plane

$$\sqrt{3}\,z + 2x - 3y = 4.$$

If d = the distance of (x_1, y_1, z_1) from $Ax + By + Cz = D$, then

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2+B^2+C^2}}.$$

Here (x_1, y_1, z_1) is $(2, -3, 0)$, and $Ax + By + Cz = D$ is

$$2x - 3y + \sqrt{3}\,z = 4;$$

hence $$d = \frac{4+9-4}{\sqrt{4+9+3}} = 2.25.$$

11. Show that the two points $(1, -1, 3)$ and $(3, 3, 3)$ are on opposite sides of, and equidistant from, the plane $5x + 2y - 7z + 9 = 0$.

If (x_1, y_1, z_1) is $(1, -1, 3)$ and $Ax + By + Cz = D$ is

$$-5x - 2y + 7z = 9,$$

$$d = \frac{-5+2+21-9}{\sqrt{25+4+49}} = \frac{9}{\sqrt{78}}.$$

If (x_1, y_1, z) is $(3, 3, 3)$, we have

$$d = \frac{-15-6+21-9}{\sqrt{25+4+49}} = \frac{-9}{\sqrt{78}}.$$

Hence the points are on opposite sides of, and equally distant from, the plane.

12. If, in Fig. 105, $OM = a$, $OR = b$, $OS = c$, find the equation of the plane through the points M, P, R. Find the length of the perpendicular from S upon this plane.

The point M is $(a, 0, 0)$, R is $(0, b, 0)$, P is (a, b, c). If the plane $Ax + By + Cz = D$ passes through each of these points, we have

$$Aa = D, \text{ or } A = D \div a,$$

$$Bb = D, \text{ or } B = D \div b,$$

$$Aa + Bb + Cc = D; \therefore C = -(D \div c).$$

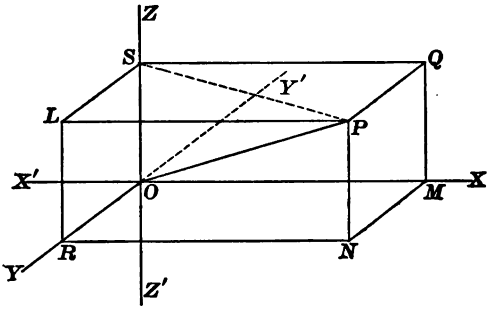

FIG. 105.

Substituting these values in $Ax + By + Cz = D$, we have

$$\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 1, \text{ or } bcx + acy - abz = abc. \qquad (1)$$

Now the distance from S, $(0, 0, c)$ to (1) is, by § 225, Cor.,

$$\frac{-2abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}.$$

13. Prove that the plane

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0, \qquad (1)$$

passes through the point (x_1, y_1, z_1) and is parallel to the plane

$$Ax + By + Cz = D. \qquad (2)$$

Plane (1) passes through the point (x_1, y_1, z_1) since its co-ordinates satisfy equation (1).

Plane (1) is parallel to plane (2), since the perpendiculars from the origin upon them have the same direction cosines and therefore coincide.

14. Find the equation of a plane passing through the point $(3, 4, -1)$, and parallel to the plane $2x + 4y - z = 2$.

Here (x_1, y_1, z_1) is $(3, 4, -1)$, $A = 2$, $B = 4$, $C = -1$.

Substituting these values in (1) of No. 13, we have

$$2(x-3) + 4(y-4) - (z+1) = 0, \text{ or } 2x + 4y - z = 23.$$

15. What three equations must be satisfied in order that the plane $Ax + By + Cz = D$ may pass through the two points (x_1, y_1, z_1), (x_2, y_2, z_2), and be perpendicular to the plane

$$A'x + B'y + C'z = D'? \qquad (1)$$

If $Ax + By + Cz = D$ passes through (x_1, y_1, z_1) and (x_2, y_2, z_2), we must have

$$Ax_1 + By_1 + Cz_1 = D, \text{ and } Ax_2 + By_2 + Cz_2 = D.$$

If $Ax + By + Cz = D$ is perpendicular to (1), by § 224, Cor. 2, we have

$$AA' + BB' + CC' = 0.$$

16. Find the equation of the plane passing through the points $(1, 1, 1)$, $(2, 0, -1)$, and perpendicular to the plane $x + y - z = 3$.

If $Ax + By + Cz = D$ passes through $(1, 1, 1)$ and $(2, 0, -1)$, we must have

$$A + B + C = D, \qquad (1)$$

$$2A - C = D. \qquad (2)$$

If $Ax + By + Cz = D$ is perpendicular to $x + y - z = 3$, we have

$$A + B - C = 0. \qquad (3)$$

Solving (1), (2), and (3) for A, B, C, we obtain

$$A = \tfrac{3}{4}D, \quad B = -\tfrac{1}{4}D, \quad C = \tfrac{1}{2}D.$$

Substituting in $Ax + By + Cz = D$, we have

$$\tfrac{3}{4}Dx - \tfrac{1}{4}Dy + \tfrac{1}{2}Dz = D, \text{ or } 3x - y + 2z = 4,$$

as the required equation.

17. What three equations must be satisfied, in order that the plane $Ax + By + Cz = D$ may pass through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)?

The three equations are evidently

$$Ax_1 + By_1 + Cz_1 = D,$$

$$Ax_2 + By_2 + Cz_2 = D,$$

$$Ax_3 + By_3 + Cz_3 = D.$$

18. Find the equation of the plane which passes through the points (1, 2, 3), (3, 2, 1), (2, 3, 1), and find the distance of this plane from the origin.

If the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ passes through these points, we have

$$\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1. \qquad (1)$$

$$\frac{3}{a}+\frac{2}{b}+\frac{1}{c}=1. \qquad (2)$$

$$\frac{2}{a}+\frac{3}{b}+\frac{1}{c}=1. \qquad (3)$$

Subtracting (1) from (2), we obtain

$$\frac{2}{a}-\frac{2}{c}=0, \text{ or } a=c.$$

Subtracting (3) from (2), we obtain

$$\frac{1}{a}-\frac{1}{b}=0, \text{ or } a=b.$$

These values in (1) give

$$\frac{1}{a}+\frac{2}{a}+\frac{3}{a}=1, \text{ or } 6=a=b=c.$$

Hence the required equation is

$$\frac{x}{6}+\frac{y}{6}+\frac{z}{6}=1, \text{ or } x+y+z=6. \qquad (1)$$

The distance of (1) from the origin is $\frac{6}{\sqrt{3}}$, or $2\sqrt{3}$.

19. Find the equation of the plane through (2, 3, −1) parallel to the plane $3x-4y+7z=0$.

Here (x_1, y_1, z_1) is (2, 3, −1), $A=3$, $B=-4$, $C=7$. Substituting these values in (1) of No. 13, we have

$$3(x-2)-4(y-3)+7(z+1)=0.$$

or
$$3x-4y+7z+13=0.$$

20. Find the equation of a plane which passes through the point (1, 2, 3), and is perpendicular to each of the planes

$$x+2z=1, \quad y+5z=1.$$

If $Ax + By + Cz = D$ passes through (1, 2, 3), we have

$$A + 2B + 3C = D. \qquad (1)$$

If $Ax + By + Cz = D$ is perpendicular to $x + 2z = 1$, we have, § 224, Cor. 2,

$$A + 2C = 0. \qquad (2)$$

If $Ax + By + Cz = D$ is perpendicular to $y + 5z = 1$, we have

$$B + 5C = 0. \qquad (3)$$

Subtracting (2) from (1), we obtain

$$2B + C = D. \qquad (4)$$

From (3) and (4),

$$C = -\tfrac{1}{9}D.$$

Hence, from (2) and (3),

$$A = \tfrac{2}{9}D, \quad B = \tfrac{5}{9}D.$$

Substituting in $Ax + By + Cz = D$, we have

$$2x + 5y - z = 9.$$

Ex. 45. Page 247.

1. Determine the position, direction cosines, and direction angles of the intersection of the planes

$$x + y - z + 1 = 0, \text{ and } 4x + y - 2z + 2 = 0.$$

Eliminating successively y and z between the equations, we obtain

$$3x - z + 1 = 0 \text{ and } 2x - y = 0; \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z-1}{3}.$$

From the last form we know that the line passes through the point (0, 0, 1), and is parallel to the radius vector of the point (1, 2, 3). The direction cosines are found by dividing the denominators 1, 2, 3, by $\sqrt{14}$; and the direction angles are

$$\cos^{-1}\frac{1}{\sqrt{14}}, \cos^{-1}\frac{2}{\sqrt{14}}, \text{ and } \cos^{-1}\frac{3}{\sqrt{14}}.$$

2. Determine the position and direction cosines of the intersection of $x - 2y = 5$, and $3x + y - 7z = 0$.

From $x - 2y = 5$ we have $\frac{x-5}{2} = \frac{y}{1}$. (1)

Eliminating x between $x - 2y = 5$ and $3x + y - 7z = 0$, we have

$$\frac{y}{1} = \frac{z - \frac{15}{7}}{1}. \qquad (2)$$

From (1) and (2) we have

$$\frac{x-5}{2}=\frac{y}{1}=\frac{z-\frac{15}{7}}{1}, \tag{3}$$

of which the locus is the line through the point $(5, 0, \frac{15}{7})$ parallel to the radius vector of (2, 1, 1).

The direction cosines are $\frac{2}{\sqrt{6}}$, $\frac{1}{\sqrt{6}}$, and $\frac{1}{\sqrt{6}}$.

3. Determine the position of the line

$$5x-4y=1,\ \ 3y-5z=2.$$

From $5x-4y=1$, we obtain $\frac{x-\frac{1}{5}}{4}=\frac{y}{5}$. (1)

From $3y-5z=2$, we obtain $\frac{y}{5}=\frac{z+\frac{2}{5}}{3}$. (2)

Hence the equations of the line are

$$\frac{x-\frac{1}{5}}{4}=\frac{y}{5}=\frac{z+\frac{2}{5}}{3},$$

the locus of which passes through $(\frac{1}{5}, 0, -\frac{2}{5})$ and is parallel to the radius vector of the point (4, 5, 3).

4. What is the position of the line $x=3$, $y=4$? Of the line $y=4$, $z=-5$? Of the line $x=-2$, $z=3$?

The line $x=3$, $y=4$ is perpendicular to the plane xy at the point (3, 4, 0). The line $y=4$, $z=-5$ is perpendicular to the plane yz at the point (0, 4, −5). The line $x=-2$, $z=3$ is perpendicular to the plane xz at the point (−2, 0, 3).

5. Find the equations of the right line passing through the points (1, 2, 3), (3, 4, 1).

The equations of the line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \tag{64}$$

Here (x_1, y_1, z_1) is (1, 2, 3) and (x_2, y_2, z_2) is (3, 4, 1); hence the required equations are

$$\frac{x-1}{3-1}=\frac{y-2}{4-2}=\frac{z-3}{1-3}, \text{ or } \frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}. \tag{1}$$

6. Find the points in which the line of No. 5 pierces the co-ordinate planes.

For $z=0$ in (1) of No. 5, $x=4$, and $y=5$; hence the line pierces the plane xy in the point (4, 5, 0).

For $x=0$, $y=1$, and $z=4$; hence the line pierces the plane yz in the point (0, 1, 4).

For $y=0$, $x=-1$, and $z=5$; hence the line pierces the plane xz in the point $(-1, 0, 5)$.

7. Two of the projecting planes of a line are

$$x+y=4 \text{ and } 2x-5z=-2;$$

find the third.

Eliminating x between $x+y=4$ and $2x-5z=-2$, we have as the equation of the third projecting plane

$$2y+5z=10.$$

8. A line passes through $(2, 1, -1)$ and $(-3, -1, 1)$; find the equations of its projections on the co-ordinate planes.

The equations of the line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \qquad [64]$$

Here (x_1, y_1, z_1) is $(2, 1, -1)$ and (x_2, y_2, z_2) is $(-3, -1, 1)$; hence the equations of the line are

$$\frac{x-2}{-3-2}=\frac{y-1}{-1-1}=\frac{z+1}{1+1},$$

or

$$\frac{x-2}{-5}=\frac{y-1}{-2}=\frac{z+1}{2}. \qquad (1)$$

From the first and second members of (1), we have

$$2x-5y+1=0. \qquad (2)$$

From the first and third members, we have

$$2x+5z+1=0. \qquad (3)$$

From the second and third members, we have

$$y+z=0. \qquad (4)$$

(2), (3), and (4) are the equations of the projections of (1) on the co-ordinate planes, xy, xz, and yz, respectively.

9. Show that the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$, and $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$ are at right angles.

The lines

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N},$$

$$\frac{x-x_2}{L'}=\frac{y-y_2}{M'}=\frac{z-z_2}{N'},$$

are at right angles if

$$LL'+MM'+NN'=0. \qquad (1)$$

Here $L=1,\ M=2,\ N=1,\ L'=1,\ M'=-1,\ N'=1.$

Substituting these values in (1) we find that the condition of perpendicularity is satisfied by the two given lines.

10. Show that the line $4x=3y=-z$ is perpendicular to the line $3x=-y=-4z$.

Writing these equations in the form of (1), § 228, we have

$$\frac{x}{\frac{1}{4}}=\frac{y}{\frac{1}{3}}=\frac{z}{-1},\quad \frac{x}{\frac{1}{3}}=\frac{y}{-1}=\frac{z}{-\frac{1}{4}}.$$

Here $L=\frac{1}{4},\ M=\frac{1}{3},\ N=-1,\ L'=\frac{1}{3},\ M'=-1,\ N'=-\frac{1}{4}.$

Hence $LL'+MM'+NN'=\frac{1}{12}-\frac{1}{3}+\frac{1}{4}=0,$

and the lines are perpendicular to each other, by § 229, Cor. 2.

11. Find the angle between the lines

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{0},\ \text{and}\ \frac{x}{3}=\frac{y}{-4}=\frac{z}{5}.$$

If θ denote the angle between the lines

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N},$$

and

$$\frac{x-x_2}{L'}=\frac{y-y_2}{M'}=\frac{z-z_2}{N'};$$

then

$$\cos\theta=\frac{LL'+MM'+NN'}{\sqrt{L^2+M^2+N^2}\,\sqrt{L'^2+M'^2+N'^2}}. \qquad [65]$$

Here $L=1,\ M=1,\ N=0,\ L'=3,\ M'=-4,\ N'=5.$

Substituting these values in [65], we obtain

$$\cos\theta=\frac{3-4}{\sqrt{2}\,\sqrt{50}}=-0.1;\ \text{hence}\ \theta=\cos^{-1}(-0.1).$$

12. Find the angle between the right lines

$$y = 5x + 3,\ z = 3x + 5,$$

and $$y = 2x,\ z = x + 1.$$

Writing these equations in the form of (1), § 228, we have

$$\frac{x}{1} = \frac{y-3}{5} = \frac{z-5}{3},$$

and $$\frac{x}{1} = \frac{y}{2} = \frac{z-1}{1}.$$

Here $L = 1$, $M = 5$, $N = 3$, $L' = 1$, $M' = 2$, $N' = 1$.

Substituting these values in [65] in No. 11, we have

$$\cos\theta = \frac{1 + 10 + 3}{\sqrt{1 + 25 + 9}\,\sqrt{1 + 4 + 1}} = \frac{14}{\sqrt{210}};$$

hence $$\theta = 14^\circ\, 57'\, 46''.$$

13. Find the angle between the lines

$$y = 2x + 2,\ z = 2x + 1,$$

and $$y = 4x + 1,\ z = x + 5.$$

Writing these equations in the form of (1), § 228, we have

$$\frac{x}{1} = \frac{y-2}{2} = \frac{z-1}{2},\ \frac{x}{1} = \frac{y-1}{4} = \frac{z-5}{1}.$$

Here $L = 1$, $M = 2$, $N = 2$, $L' = 1$, $M' = 4$, $N' = 1$.

Substituting these values in [65] in No. 11, we have

$$\cos\theta = \frac{1 + 8 + 2}{\sqrt{1 + 4 + 4}\,\sqrt{1 + 16 + 1}} = \frac{11}{9\sqrt{2}} = \tfrac{11}{18}\sqrt{2};$$

hence $$\theta = \cos^{-1}\tfrac{11}{18}\sqrt{2}.$$

14. Show that the lines

$$3x + 2y + z - 5 = 0,\ x + y - 2z - 3 = 0,$$

and $$8x - 4y - 4z = 0,\ 7x + 10y - 8z = 0,$$

are at right angles.

Eliminating successively x and y between each pair of these equations, we obtain

$$y - 7z = 4,\ x + 5z = -1,\ \text{or}\ \frac{x+1}{-5} = \frac{y-4}{7} = \frac{z}{1},$$

and $$3y = z,\ 3x = 2z,\ \text{or}\ \frac{x}{2} = \frac{y}{1} = \frac{z}{3}.$$

Here $L=-5,\ M=7,\ N=1,\ L'=2,\ M'=1,\ N'=3.$

Hence $LL'+MM'+NN'=-10+7+3=0,$

and the two lines are at right angles, by § 229, Cor. 2.

15. Find the equations of the line through $(-2, 3, -1)$ parallel to the line $y=-2x+1,\ z=3x-4$.

Putting these equations in the form of (1), § 228, we have

$$\frac{x}{1}=\frac{y-1}{-2}=\frac{z+4}{3}. \tag{1}$$

Hence the required equations are

$$\frac{x+2}{1}=\frac{y-3}{-2}=\frac{z+1}{3}. \tag{2}$$

For (2) is parallel to (1), since it has the same direction cosines; and it passes through $(-2, 3, -1)$, as the co-ordinates of this point satisfy (2).

16. Find the equations of the right line through $(3, -7, -5)$, its direction cosines being proportional to $-3, 5, -6$.

The equations of the line through (x_1, y_1, z_1), and having direction cosines proportional to L, M, and N are

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N}. \tag{1}$$

Here (x_1, y_1, z_1) is $(3, -7, -5)$ and L, M, N are $-3, 5, -6$ respectively; hence the required equations are

$$\frac{x-3}{-3}=\frac{y+7}{5}=\frac{z+5}{-6}.$$

17. Find the equations of the line through $(2, -4, -6)$ perpendicular to the plane $3x-6y+2z=4$.

By § 230, Cor. 2, the line

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N} \tag{1}$$

is perpendicular to the plane $Ax+By+Cz=D$ if

$$\frac{L}{A}=\frac{M}{B}=\frac{N}{C}. \tag{2}$$

Here (x_1, y_1, z_1) is $(2, -4, -6)$, $A=3$, $B=-6$, $C=2$.

Substituting these values of A, B, C in (2), we obtain

$$M = -2L, \quad N = \tfrac{2}{3}L.$$

Substituting these values of M and N, and those of x_1, y_1, z_1 in (1), we obtain as the required equations

$$\frac{x-2}{3} = \frac{y+4}{-6} = \frac{z+6}{2}.$$

18. Find the inclination of the line

$$\frac{x-4}{3} = \frac{y+2}{-2} = \frac{z-5}{-4}$$

to the plane $2x - 4y + 3z = 1$.

If v denote the inclination of the line

$$\frac{x-x_1}{L} = \frac{y-y_1}{M} = \frac{z-z_1}{N}$$

to the plane $Ax + By + Cz = D$;

then
$$\sin v = \frac{AL + BM + CN}{\sqrt{A^2 + B^2 + C^2}\sqrt{L^2 + M^2 + N^2}}. \qquad [66]$$

Here $L = 3$, $M = -2$, $N = -4$, $A = 2$, $B = -4$, $C = 3$.

Substituting these values in [66], we obtain

$$\sin v = \frac{6 + 8 - 12}{\sqrt{4 + 16 + 9}\sqrt{9 + 4 + 16}} = \frac{2}{29},$$

or
$$v = \sin^{-1}\tfrac{2}{29}.$$

19. Reduce the equations $x = mz + p$, $y = nz + q$, to the symmetrical form, and thus find the direction cosines in terms of m and n.

Reducing these equations to the form of (1), § 228, we have

$$\frac{x-p}{m} = \frac{y-q}{n} = \frac{z}{1}. \qquad (1)$$

Dividing each denominator by $\sqrt{m^2 + n^2 + 1}$, we obtain the symmetrical form

$$\frac{x-p}{\dfrac{m}{\sqrt{m^2+n^2+1}}} = \frac{y-q}{\dfrac{n}{\sqrt{m^2+n^2+1}}} = \frac{z}{\dfrac{1}{\sqrt{m^2+n^2+1}}},$$

in which the denominators equal $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively.

20. Show that the formula for the angle included between the lines

$$x = mz + p,\quad y = nz + q,$$

and $$x = m'z + p',\quad y = n'z + q',$$

is $$\cos\theta = \frac{mm' + nn' + 1}{\sqrt{m^2 + n^2 + 1}\,\sqrt{m'^2 + n'^2 + 1}}.$$

By No. 19, the direction cosines of the first line are

$$\frac{m}{\sqrt{m^2 + n^2 + 1}},\quad \frac{n}{\sqrt{m^2 + n^2 + 1}},\quad \frac{1}{\sqrt{m^2 + n^2 + 1}};$$

and those of the second are

$$\frac{m'}{\sqrt{m'^2 + n'^2 + 1}},\quad \frac{n}{\sqrt{m'^2 + n'^2 + 1}},\quad \frac{1}{\sqrt{m'^2 + n'^2 + 1}}.$$

Substituting these values in the formula

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma',$$

we have $$\cos\theta = \frac{mm' + nn' + 1}{\sqrt{m^2 + n^2 + 1}\,\sqrt{m'^2 + n'^2 + 1}}. \qquad (1)$$

21. Prove that the lines in No. 20 are perpendicular if

$$mm' + nn' + 1 = 0,$$

and conversely. Prove that they are parallel if $m = m'$ and $n = n'$, and conversely.

If, in (1) of No. 20, $mm' + nn' + 1 = 0$, $\cos\theta = 0$, and $\theta = 90$; that is, the lines are at right angles.

Conversely, if the lines are at right angles,

$$\cos\theta = 0;\ \therefore\ mm' + nn' + 1 = 0.$$

If $m = m'$ and $n = n'$, $\cos\theta = 1$, and $\theta = 0°$; that is, the lines are parallel.

Conversely, if the lines are parallel, their projecting planes on either co-ordinate plane are parallel, and hence their projections on either one of these planes are parallel; therefore the slopes of these projections are equal; that is, $m = m'$ and $n = n'$.

22. Prove that two lines are parallel if their projections are parallel, and conversely.

If the projections of two lines on the co-ordinate planes are parallel, the slopes of these projections are equal; that is, $m = m'$ and $n = n'$. Hence the lines are parallel by No. 21. The converse is proved in No. 21.

Ex. 46. Page 263.

1. What is locus in space of

$$x^3 + 3x^2 - 6x - 8 = 0?$$
$$y^3 - 2y^2 - 5y + 6 = 0?$$
$$z^2 + mz = 0?$$

The equation $x^3 + 3x^2 - 6x - 8 = 0$ may be written in the form

$$(x + 4)(x + 1)(x - 2) = 0,$$

of which the locus in space is the planes $x = -4$, $x = -1$, and $x = 2$ [§ 236 (i.)].

The equation $y^3 - 2y^2 - 5y + 6 = 0$ may be written in the form

$$(y + 2)(y - 1)(y - 3) = 0,$$

of which the locus in space is the planes $y = -2$, $y = 1$, and $y = 3$.

The equation $z^2 + mz = 0$ may be written in the form

$$z(z + m) = 0,$$

of which the locus is the planes $z = 0$ and $z = -m$.

2. What is the locus in space of

$$y^2 = 8x? \quad 4x^2 + 9y^2 = 36?$$
$$9z^2 - 16y^2 = 144? \quad z^2 + x^2 = r^2?$$
$$(2a - z)(y^2 - b^2) = 0?$$

The locus in space of $y^2 = 8x$ is the parabolic cylinder whose elements are parallel to the axis of z, and whose trace on the plane xy is the parabola $y^2 = 8x$ [§ 236 (ii.)].

The locus of $4x^2 + 9y^2 = 36$ is the elliptical cylinder whose elements are parallel to the axis of z, and whose trace on the plane xy is the ellipse $4x^2 + 9y^2 = 36$.

The locus of $9z^2 - 16y^2 = 144$ is the hyperbolic cylinder whose elements are parallel to the axis of x and whose trace on the plane yz is the hyperbola $9z^2 - 16y^2 = 144$.

The locus of $z^2 + x^2 = r^2$ is the circular cylinder whose elements are parallel to the axis of y; and whose trace on the plane zx is the circle

$$z^2 + x^2 = r^2.$$

The equation $(2a - z)(y^2 - b^2)$ may be written in the form

$$(2a - z)(y + b)(y - b) = 0,$$

of which the locus is the planes $z = 2a$, $y = -b$, and $y = b$.

3. Find the equations of the projecting cylinders of the curve

$$x^2 + 3y^2 - 2z^2 = 8, \quad x^2 + 2y^2 + 3z^2 = 16.$$

Eliminating z between the given equations of the curve, we obtain

$$5x^2 + 13y^2 = 56$$

as the equation of the projecting cylinder of the curve upon the plane xy.

Eliminating x between these equations we obtain

$$5z^2 - y^2 = 8$$

as the equation of the projecting cylinder of the curve upon the plane yz.

Eliminating y we find the equation of the third projecting cylinder to be

$$x^2 + 13z^2 = 32.$$

4. Find the equations of the projections of the curves

$$x^2 + y^2 + 2z^2 = 16, \quad 9(x^2 + y^2) + 4z^2 = 36.$$

Eliminating z between the equations of the curves we find that the equation of their projecting cylinder on the plane xy is $x^2 + y^2 = \frac{4}{7}$, which is also the equation of their projections on the plane xy.

Eliminating x or y we find that $z = \pm \frac{3}{7}\sqrt{42}$ are the projecting planes of the curves on the plane yz or zx. Hence the planes of the curves are parallel to the plane xy, and therefore the curves are equal to their projections on this plane; that is, they are two circles in the planes $z = \pm \frac{3}{7}\sqrt{42}$, whose centres are in the axis of z and whose radii are $\sqrt{\frac{4}{7}}$ each.

5. Find the semi-axes and eccentricity of the ellipse

$$4x^2 + 9y^2 + 4z^2 = 37, \quad z = \tfrac{1}{2}.$$

Eliminating z between the equations we find that the equation of the projection of the curve on the plane xy is

$$4x^2 + 9y^2 = 36. \qquad (1)$$

But, since the curve is in the plane $z = \frac{1}{2}$, its plane is parallel to the plane xy, and therefore the curve itself is equal to its projection (1).

Hence $a = 3, \; b = 2, \text{ and } e = \frac{1}{3}\sqrt{5}.$

6. Find the nature of the curves

$$x^2 + y^2 + 4z^2 = 25, \quad 7(x^2 + y^2) - 4z^2 = 79.$$

Eliminating x or y between the two equations we find that the curves lie in the planes $z = \pm 3$, and are therefore parallel to the plane xy, and equal to their projections on this plane.

Eliminating z between the equations of the curves, we obtain

$$x^2 + y^2 = 13$$

as the equation of the projections of the curves on the plane xy; hence the curves are two circles whose centres are in the axis of z and whose radii are $\sqrt{13}$.

7. Find the traces of the surface $2x^2 + 5y^2 - 7z^2 = 9$; of the surface $x^2 + 3y^2 = 8z$.

Putting $z = 0$ in $2x^2 + 5y^2 - 7z^2 = 9$, we obtain $2x^2 + 5y^2 = 9$ as the equation of the trace of the surface on the plane xy, which is therefore an ellipse.

The trace on the plane xz is the hyperbola $2x^2 - 7z^2 = 9$, and the trace on the plane yz is the hyperbola $5y^2 - 7z^2 = 9$.

The trace of $x^2 + 3y^2 = 8z$ on the plane xy is $x^2 + 3y^2 = 0$ or the point (0, 0); the trace on the plane xz is the parabola $x^2 = 8z$, and the trace on the plane yz is the parabola $y^2 = \frac{8}{3}z$. The axis of the parabola in either case lies on the axis of z.

8. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $z = \pm 3x + 5$; find its trace on the plane xy.

Here the meridian section in the plane xz is

$$z = \pm 3x + 5, \text{ or } x^2 = \tfrac{1}{9}(z - 5)^2.$$

Hence $$r^2 = f(z) = \tfrac{1}{9}(z - 5)^2. \qquad (\S\ 239)$$

Substituting in [67], we obtain as the required equation

$$x^2 + y^2 = \tfrac{1}{9}(z - 5)^2,$$

or $$9(x^2 + y^2) = (z - 5)^2, \qquad (1)$$

of which the trace on the plane xy is the circle $x^2 + y^2 = \frac{25}{9}$.

9. Find the equation of a cone of revolution one of whose traces is $x^2 + y^2 = 9$, and whose vertex is (0, 0, 5).

Since the vertex is (0, 0, 5), c of [73] equals 5; and since the radius of the trace on the plane xy equals 3, m of [73] equals $\frac{5}{3}$. Substituting these values in

$$m^2(x^2 + y^2) = (z - 3)^2, \qquad [73]$$

we obtain as the required equation

$$\tfrac{25}{9}(x^2 + y^2) = (z - 5)^2.$$

or $$25x^2 + 25y^2 - 9z^2 + 90z = 225.$$

10. Find the equation of the paraboloid of revolution one of whose traces is $2x^2 = 3z + 5$.

This trace is a meridian section and the axis of z is the axis of the paraboloid. Hence $r^2 = f(z) = \frac{3}{2}z + \frac{5}{2}$. Substituting in [67], we obtain as the required equation

$$x^2 + y^2 - \tfrac{3}{2}z - \tfrac{5}{2} = 0.$$

11. Find the equation of the paraboloid of revolution one of whose traces is $y^2 = 8x$.

This trace is a meridian section, and the axis of x is the axis of the paraboloid. By § 239 we know that, if the axis of x is the axis of revolution and the meridian section be taken in the plane xy, the general equation of a surface of revolution is

$$y^2 + z^2 = f(x), \qquad (1)$$

in which $f(x) = r^2 = y^2$ in the equation of the meridian section.

Here $\qquad y^2 = f(x) = 8x.$

Substituting in (1), we obtain

$$y^2 + z^2 - 8x = 0.$$

12. Find the equation of the cone of revolution whose axis is the axis of z, and one of whose traces is $2y = \pm z + 6$; find its vertex.

If the trace on the plane yz is $2y = \pm z + 6$, the meridian section in plane xz is $2x = \pm z + 6$, or the part that inclines to the right is

$$2x = z + 6, \text{ or } z = 2x - 6. \qquad (1)$$

Comparing (1) with $z = mx + c$, we have $m = 2$ and $c = -6$. Making these substitutions in [73], we obtain

$$4(x^2 + y^2) = (z + 6)^2,$$

or $\qquad x^2 + y^2 - \frac{1}{4}z^2 - 3z = 9.$

Or, from (1),

$$x^2 = \tfrac{1}{4}(z + 6)^2 = f(z).$$

Substituting in [67], we obtain

$$x^2 + y^2 = \tfrac{1}{4}(z + 6)^2, \text{ or } x^2 + y^2 - \tfrac{1}{4}z^2 - 3z = 9.$$

13. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $9x^2 + 4z^2 = 36$.

Since the meridian section in the plane xz is

$$9x^2 + 4z^2 = 36, \text{ or } x^2 = \tfrac{1}{9}(36 - 4z^2),$$

we have r^2, or $f(z) = \frac{1}{9}(36 - 4z^2)$.

Substituting in [67], we have

$$x^2 + y^2 = \tfrac{1}{9}(36 - 4z^2), \text{ or } 9(x^2 + y^2) + 4z^2 = 36,$$

as the required equation.

14. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $16y^2 + 9z^2 = 144$.

Since the trace on the plane yz is

$$16y^2 + 9z^2 = 144,$$

the meridian section in the plane xz is

$$16x^2 + 9z^2 = 144, \text{ or } x^2 = \tfrac{1}{16}(144 - 9z^2).$$

Hence $f(z) = \frac{1}{16}(144 - 9z^2)$.

Substituting in [67], we have

$$x^2 + y^2 = \tfrac{1}{16}(144 - 9z^2), \text{ or } 16(x^2 + y^2) + 9z^2 = 144.$$

15. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $9z^2 - 4y^2 = -36$.

Since the trace on the plane yz is

$$9z^2 - 4y^2 = -36,$$

the meridian section in the plane xz is

$$9z^2 - 4x^2 = -36, \text{ or } x^2 = \tfrac{1}{4}(9z^2 + 36).$$

Hence $f(z) = \frac{1}{4}(9z^2 + 36)$.

Substituting in [67], we have

$$x^2 + y^2 = \tfrac{1}{4}(9z^2 + 36), \text{ or } 4(x^2 + y^2) - 9z^2 = 36.$$

16. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $z^2x = 1$; also when one trace is $z^8 = 2y^2$.

(i.) If $z^2x = 1$, $x^2 = \frac{1}{z^4}$; hence $f(z) = \frac{1}{z^4}$. Substituting in [67], we have

$$x^2 + y^2 = \frac{1}{z^4}, \text{ or } x^2z^4 + y^2z^4 = 1,$$

as the first required equation.

(ii.) If the trace on the plane yz is $z^3 = 2y^2$, the meridian section in the plane xz is $z^3 = 2x^2$.

Hence $x^2 = \frac{1}{2}z^3; \therefore f(z) = \frac{1}{2}z^3.$

Substituting in [67], we have

$$x^2 + y^2 = \tfrac{1}{2}z^3, \text{ or } 2(x^2 + y^2) = z^3,$$

as the required equation.

17. Each element of a cone makes an angle of 45° with its axis; find the semi-axes of the section made by a plane cutting the axis 5 below the vertex and at an angle of 60°.

Take the axis of the cone as the axis of z, and the plane perpendicular to this axis 5 below the vertex as the plane xy, and the intersection of this plane with the plane cutting the cone as the axis of y. Then the equation of the section is

$$y^2 \tan^2\theta + x^2\cos^2\theta(\tan^2\theta - \tan^2\phi) + 2cx\sin\phi - c^2 = 0, \qquad [74]$$

in which θ is 45°, and $\phi = 30°$; for if an element makes an angle of 45° with the axis, it will make the same angle with the axis of x; and if the cutting plane makes an angle of 60° with the axis of z, it will make an angle of 30° with the plane xy; also $c = 5$. Substituting these values of c, θ, and ϕ in [74], we have

$$y^2 + x^2(\tfrac{3}{4})(1 - \tfrac{1}{3}) + 5x - 25 = 0,$$

or $$y^2 + \tfrac{1}{2}x^2 + 5x = 25, \text{ or } y^2 + \tfrac{1}{2}(x + 5)^2 = \tfrac{75}{2}. \qquad (1)$$

Passing to parallel axes through $(-5, 0)$, (1) becomes

$$y^2 + \frac{x^2}{2} = \frac{75}{2}, \text{ or } \frac{x^2}{75} + \frac{y^2}{\frac{75}{2}} = 1. \qquad (2)$$

The locus of (2) is an ellipse, whose semi-axes are $5\sqrt{3}$ and $\frac{5}{2}\sqrt{6}$.